ANISOTROPIC ELLIPTIC OPTICAL FIBERS

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THESIS

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SUMMARY

The instability of the fabrication process of optical fibers introduces both ellipticies and stress anisotropy. These perturbations are the causes for birefringence in single-mode optical fibers which have been researched extensively. In this research, the mode propagations in optical fibers with anisotropic elliptical core have been investigated.

The exact characteristic equation for anisotropic elliptical optical fibers can be obtained for odd and even hybrid modes in terms of infinite determinants utilizing Mathieu and modified Mathieu functions. The exact characteristic equation is applicable to elliptical fibers that have any ellipticity. A simplified characteristic equation can then be obtained by applying the weakly guiding approximation such that the difference in the refractive indices of the core and the cladding is small. It has been shown that significant simplification can be achieved under this approximation.

The simplified characteristic equation is used to compute the propagation constants for the anisotropic elliptical fiber. The expression for the power carried by the fiber is also obtained and numerical results are presented. These results may be used to approximate a number of different shapes of fibers.

1. INTRODUCTION

The circular optical fiber is one of the most studied media for long distance communication. An optical fiber consists of a core of a dielectric material in which the refractive index is higher than the refractive index of the cladding. However, a cladded fiber is often times modeled as a dielectric rod when the cladding radius is large enough such that the guided mode fields will decay to insignificant values at the outer boundary of cladding. The theory of optical fibers of this type is well understood and has been described in detail in the previously published research[1,2].

In an effort to obtain a low-loss fiber, Monerie[3] carried out an experimental study of doubly clad fibers in which the refractive index of inner cladding was less than that of core and outer cladding. This study shows that the optimum doping levels in the core of doubly clad fibers are less than those required by dispersion-free singly clad fibers. This leads to a smaller propagation loss, since the scattering losses decreased with a decreasing dopant concentration in fibers.

It is interesting to note, however, that the instability in the fabrication process may introduce ellipticities in the optical fibers. This lack of circular symmetry is one of the causes for birefringence in single-mode optical fibers; such a birefringent fiber is also called a single-polarization single-mode fiber[4]. These birefringent fibers are important for systems utilizing such fibers as fiber optic sensors and for predicting the transmission bandwidth reduction caused by group-delay differences between orthogonally polarized modes.

The birefringence due to ellipticity has been studied experimentally[5,6] and the measured data have been compared with those

equation for an uniaxially anisotropic circular rod for hybrid modes of excitation. Analytic solutions for the circular fiber when both core and cladding consist of uniaxial material was presented by Tonning[32]. This study indicates that the cut-off frequency for the lowest-order mode is not affected by the cladding anisotropy.

When the circular cross-section of the fiber is deformed into a noncircularly symmetric profile, a single mode in a circular fiber may split into two modes with different polarizations and propagation velocities[33]. This has been experimentally verified by employing the near-field method[34] and the spectral polarization method[35]. Cozen and Dyott[36] obtained the cut-off frequency of the first higher order mode in an elliptical fiber from an approximate characteristic equation. However, the limitation of their results is described by Citerne[37] and Rengarajan[38]. The cut-off characteristic has also been obtained by solving the exact characteristic equation in terms of Mathieu functions and modified Mathieu functions[38,39] and by applying the mode-matching method[27] or the critical wavelength shift formular method[40]. However, there exists a disagreement in the interpretation of their results, especially in the region where the ratio between minor axis and major axis is small; Saad[41] presented possible reasons for these differences.

For most of the practical fibers used as optical communication lines, the simplification of the characteristic equation is possible by applying the weakly guiding approximation such that the difference in the refractive indices of the core and the cladding is small[26,33,42,43]. It is shown that the error introduced by this simplification is less than 10% even when the difference in the refractive indices is equal to two. The perturbation method

can also be applied to study the polarization effects in multi-moded fibers when the fiber is weakly guiding and/or weakly anisotropic[44]. It is also possible for multi-mode fibers to simplify the characteristic equation by applying a perturbation method based on the far-from-cut-off approximation as shown by Paul and Shevgaonkar[45] in a study of circular fiber with uniaxial anisotropy. This approximation is useful for the multi-mode propagation in optical waveguides since the lower order modes that carry most of the power could be considered in the far-from-cut-off region.

For the more general case of biaxial anisotropic waveguide, an analytic solution of the field equations is not possible even for waveguides with simple geometries. However, the fields and propagation constants can be obtained by applying the numerical techniques discussed above. The propagation constants can also be computed by using a coupled mode theory. This coupled mode approch has been applied for the study of mode propagation in rectangular guides[46] and cylindrical fibers[47].

As it has been shown through the previous discussion, the instability of the fabrication process of optical fibers introduces both ellipticities and stress anisotropy. Also, the results obtained for an elliptical optical fiber may be used to approximate a number of different shapes of fibers. It can take the shape of a circular fiber or that of a flat tape fiber depending upon the eccentricity of the elliptical fiber. Hence, it is proposed to investigate the mode propagation in elliptical optical fibers containing uniaxial anisotropic media. In this study, the fiber will be modeled as a dielectric elliptical rod, since the departure of the cladding's cross-section from circular form can be ignored in the case of large dimension of cladding radius. The exact characteristic equation for the anisotropic elliptical

fiber having any ellipticity will be obtained using the series of Mathieu and modified Mathieu functions. A simplified characteristic equation will then be obtained by applying the weakly guiding approximation and the computed results will be presented.

2. WAVE EQUATION IN ELLEIPTICAL COORDINATES

In solving Maxwell's equations, the wave equation in the waveguide can be expressed in the orthogonal curvilinear coordinates (φ_1, φ_2, z) as

$$(1/\ell_1^2)(\partial^2 E_z/\partial\ell_1^2) + (1/\ell_2^2)(\partial^2 E_z/\partial\varphi_2^2)$$

$$+ (1/\ell_1\ell_2)\{(\partial(\ell_2/\ell_1)/\partial\varphi_1)(\partial E_z/\partial\varphi_1) + (\partial(\ell_1/\ell_2)/\partial\varphi_2)(\partial E_z/\partial\varphi_2)\} \tag{2.1}$$

$$+ k_1^2 E_z = 0$$

where k_1 is a constant and ℓ_1 and ℓ_2 are multiplying factors depending upon the particular coordinates[48]. $\partial/\partial z$ is replaced by $-i\beta$. An identical equation can be obtained for H_z.

For the elliptical coordinates shown in Figure 1,

$$\varphi_1 = \xi, \quad \varphi_2 = \eta \tag{2.2}$$

and

$$\ell_1 = \ell_2 = q\,[(\cosh 2\xi - \cos 2\eta)/2]^{½}. \tag{2.3}$$

By substituting Eqs.(2.2) and (2.3) into Eq.(2.1), the following equation is obtained

$$\partial^2 E_z/\partial\xi^2 + \partial^2 E_z/\partial\eta^2 + 2k^2(\cosh 2\xi - \cos 2\eta)E_z = 0 \tag{2.4}$$

with $2k = k_1 q$. Then Eq.(2.4) is the two-dimentional wave equation in elliptical coordinates.

If we let the solution of Eq.(2.4) be $E_z(\xi, \eta) = \Psi(\xi)\Phi(\eta)$, Eq.(2.4) becomes

$$\Phi d^2\Psi/d\xi^2 + \Psi d^2\Phi/d\eta^2 + 2k^2(\cosh 2\xi - \cos 2\eta)\Psi\Phi = 0 \tag{2.5}$$

Dividing Eq.(2.5) by $\Psi\Phi$ yields

$$(1/\Psi\, d^2\Psi/d\xi^2 + 2k^2\cosh 2\xi) + (1/\Phi\, d^2\Phi/d\eta^2 - 2k^2\cos 2\eta) = 0. \tag{2.6}$$

Since the equations in the parenthis are independent to each other, we obtain

$$d^2\Phi/d\eta^2 + (a - 2k^2\cos 2\eta)\Phi = 0 \tag{2.7}$$

and

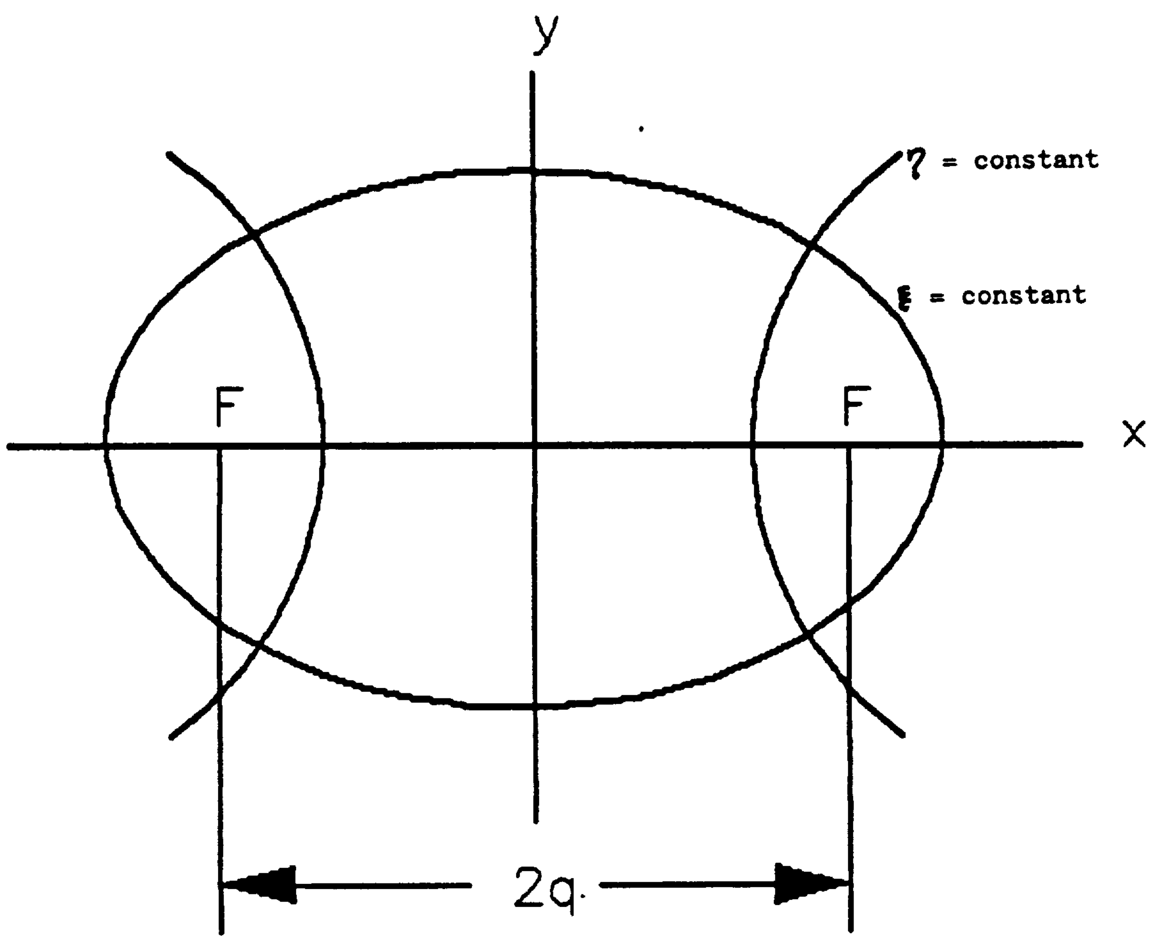


Figure 1. Elliptical coordinate system

$$(2.8) \qquad d^2\Psi/d\xi^2 - (a - 2k^2\cosh 2\xi)\Psi = 0$$

where a is the separation constant. Then the solutions for Eq.(2.4) are

$$(2.9) \qquad E_z = \left\{ \begin{array}{ll} Ce_m(\xi, k^2)ce_m(\eta, k^2) & \text{(even)} \\ Se_m(\xi, k^2)se_m(\eta, k^2) & \text{(odd)} \end{array} \right\}$$

for $k^2 > 0$

$$(2.10) \qquad E_z = \left\{ \begin{array}{ll} Fek_m(\xi, k^2)ce_m(\eta, k^2) & \text{(even)} \\ Gek_m(\xi, k^2)se_m(\eta, k^2) & \text{(odd)} \end{array} \right\}$$

for $k^2 < 0$.

Similary, the solutions for H_z can be obtained using the method discussed above.

3. CHARACTERISTIC EQUATION

The geometry shown in Figure 1 consists of an uniaxialy anisotropic elliptical rod with a permittivity tensor

$$\varepsilon_1 = \begin{vmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_1 & 0 \\ 0 & 0 & g\varepsilon_1 \end{vmatrix} \tag{3.1}$$

which is embedded in a lossless dielectric medium of permittivity ϵ_0. The anisotropic parameter g indicates the effect of anisotropic dielectric. The anisotropic parameter is unity for isotropic case.

It has been shown that in order to satisfy the boundary conditions completely both longitudinal electric and magnetic fields must be present, thus only hybrid type modes exist in elliptical fibers[2]. Furthermore, due to the asymmetry of the elliptical cylinder, two types of modes exist and they are designated as an even type modes and an odd type modes.

3.1 EVEN MODES

Assuming the t-z dependence of $e^{i(wt-\beta z)}$ for all field compornents, where β is the propagation constant and w is the angular frequency, the axial components of the field for even modes are

$$\begin{aligned} E_{z1} &= \sum_{m=1}^{\infty} A_{1m} Se_m(\xi, \gamma_{1e}^2) se_m(\eta, \gamma_{1e}^2) \\ H_{z1} &= \sum_{m=0}^{\infty} B_{1m} Ce_m(\xi, \gamma_{1h}^2) ce_m(\eta, \gamma_{1h}^2) \end{aligned} \tag{3.2}$$

for $0 \le \xi \le \xi_0$

$$\begin{aligned} E_{z2} &= \sum_{m=1}^{\infty} A_{2m} Gek_m(\xi, \gamma_2^2) se_m(\eta, \gamma_2^2) \\ H_{z2} &= \sum_{m=0}^{\infty} B_{2m} Fek_m(\xi, \gamma_2^2) ce_m(\eta, \gamma_2^2) \end{aligned} \tag{3.3}$$

for $\xi_0 \le \xi < \infty$

where A_{im} and B_{im}, $i = 1,2$ are arbitrary constants, and

$$\gamma_{1e}^2 = q^2/4(v^2\mu g\epsilon_1 - g\beta^2)$$

(3.4) $$\gamma_{1h}^2 = q^2/4(v^2\mu\epsilon_1 - \beta^2)$$

$$\gamma_2^2 = q2/4(v^2\mu\epsilon_o - \beta^2)$$

q is the semifocal length of the ellipse and μ is the permeability.

The transverse field components are

for $0 \le \xi \le \xi_o$

(3.5) $$E_{\xi 1} = -i/(v^2\mu\epsilon_1 - \beta^2)L$$
$$[\ \beta \sum_{m=1}^{\infty} A_{1m}Se_m'(\xi,\ \gamma_{1e}^2)se_m(\eta,\ \gamma_{1e}^2) + v\mu \sum_{m=0}^{\infty} B_{1m}Ce_m(\xi,\ \gamma_{1h}^2)ce_m'(\eta,\ \gamma_{1h}^2)]$$

(3.6) $$E_{\eta 1} = -i/(v^2\mu\epsilon_1 - \beta^2)L$$
$$[\ \beta \sum_{m=1}^{\infty} A_{1m}Se_m(\xi,\ \gamma_{1e}^2)se_m'(\eta,\ \gamma_{1e}^2) - v\mu \sum_{m=0}^{\infty} B_{1m}Ce_m'(\xi,\ \gamma_{1h}^2)ce_m(\eta,\ \gamma_{1h}^2)]$$

(3.7) $$H_{\xi 1} = -i/(v^2\mu\epsilon_1 - \beta^2)L$$
$$[\ -v\epsilon_1 \sum_{m=1}^{\infty} A_{1m}Se_m(\xi,\ \gamma_{1e}^2)se_m'(\eta,\ \gamma_{1e}^2) + \beta \sum_{m=0}^{\infty} B_{1m}Ce_m'(\xi,\ \gamma_{1h}^2)ce_m(\eta,\ \gamma_{1h}^2)]$$

(3.8) $$H_{\eta 1} = -i/(v^2\mu\epsilon_1 - \beta^2)L$$
$$[\ v\epsilon_1 \sum_{m=1}^{\infty} A_{1m}Se_m'(\xi,\ \gamma_{1e}^2)se_m(\eta,\ \gamma_{1e}^2) + \beta \sum_{m=0}^{\infty} B_{1m}Ce_m(\xi,\ \gamma_{1h}^2)ce_m'(\eta,\ \gamma_{1h}^2)]$$

for $\xi_o \le \xi < \infty$

(3.9) $$E_{\xi 2} = -i/(v^2\mu\epsilon_o - \beta^2)L$$
$$[\ \beta \sum_{m=1}^{\infty} A_{2m}Gek_m'(\xi,\ \gamma_2^2)se_m(\eta,\ \gamma_2^2) + v\mu \sum_{m=0}^{\infty} B_{2m}Fek_m(\xi,\ \gamma_2^2)ce_m'(\eta,\ \gamma_2^2)]$$

(3.10) $$E_{\eta 2} = -i/(v^2\mu\epsilon_o - \beta^2)L$$
$$[\ \beta \sum_{m=1}^{\infty} A_{2m}Gek_m(\xi,\ \gamma_2^2)se_m'(\eta,\ \gamma_2^2) - v\mu \sum_{m=0}^{\infty} B_{2m}Fek_m'(\xi,\ \gamma_2^2)ce_m(\eta,\ \gamma_2^2)]$$

$$(3.11) \qquad H_{\xi 2} = -i/(v^2\mu\epsilon_0 - \beta^2)L$$
$$[\ -v\epsilon_0 \sum_{m=1}^{\infty} A_{2m} Gek_m(\xi, \gamma_2^2) se_m'(\eta, \gamma_2^2)$$
$$+ \beta \sum_{m=0}^{\infty} B_{2m} Fek_m'(\xi, \gamma_2^2) ce_m(\eta, \gamma_2^2)]$$

$$(3.12) \qquad H_{\eta 2} = -i/(v^2\mu\epsilon_0 - \beta^2)L$$
$$[\ v\epsilon_0 \sum_{m=1}^{\infty} A_{2m} Gek_m'(\xi, \gamma_2^2) se_m(\eta, \gamma_2^2)$$
$$+ \beta \sum_{m=0}^{\infty} B_{2m} Fek_m(\xi, \gamma_2^2) ce_m'(\eta, \gamma_2^2)]$$

where

$$L = q[(\cosh 2\xi - \cos 2\eta)/2]^{1/2}$$

and the derivative with respect to ξ or η is denoted by the prime.

The boundary conditions require that the tangential E and H fields be continuous at the dielectric discontinuities. Equating the tangential fields at the boundary surface, $\xi = \xi_0$, gives

$$(3.13) \qquad \sum_{m=1}^{\infty} A_{1m} Se_m(\xi_0) se_m(\eta) = \sum_{m=1}^{\infty} A_{2m} Gek_m(\xi_0) se_m^*(\eta)$$

$$(3.14) \qquad \sum_{m=0}^{\infty} B_{1m} Ce_m(\xi_0) ce_m(\eta) = \sum_{m=0}^{\infty} B_{2m} Fek_m(\xi_0) ce_m^*(\eta)$$

$$(3.15) \qquad 1/(v^2\mu\epsilon_1 - \beta^2)\ [\ \beta \sum_{m=1}^{\infty} A_{1m} Se_m(\xi_0) se_m'(\eta)$$
$$- v\mu \sum_{m=0}^{\infty} B_{1m} Ce_m'(\xi_0) ce_m(\eta)]$$
$$= 1/(v^2\mu\epsilon_0 - \beta^2)\ [\ \beta \sum_{m=1}^{\infty} A_{2m} Gek_m(\xi_0) se_m^{*\prime}(\eta)$$
$$- v\mu \sum_{m=0}^{\infty} B_{2m} Fek_m'(\xi_0) ce_m^*(\eta)]$$

$$(3.16) \qquad 1/(v^2\mu\epsilon_1 - \beta^2)\ [\ v\epsilon_1 \sum_{m=1}^{\infty} A_{1m} Se_m'(\xi_0) se_m(\eta)$$
$$+ \beta \sum_{m=0}^{\infty} B_{1m} Ce_m(\xi_0) ce_m'(\eta)]$$
$$= 1/(v^2\mu\epsilon_0 - \beta^2)\ [\ v\epsilon_0 \sum_{m=1}^{\infty} A_{2m} Gek_m'(\xi_0) se_m^*(\eta)$$
$$+ \beta \sum_{m=0}^{\infty} B_{2m} Fek_m(\xi_0) ce_m^{*\prime}(\eta)]$$

where the following abbreviations have been used,

$$(3.17) \qquad Se_m(\xi_0) = Se_m(\xi_0, \gamma_{1e}^2)$$

$$(3.18) \qquad se_m(\eta) = se_m(\eta, \gamma_{1e}^2)$$

$$(3.19) \qquad Ce_m(\xi_0) = Ce_m(\xi_0, \gamma_{1h}^2)$$

$$(3.20) \qquad ce_m(\eta) = ce_m(\eta, \gamma_{1h}^2)$$

(3.21) $\quad Gek_m(\xi_0) = Gek_m(\xi_0, \gamma_2^2)$

(3.22) $\quad se_m^*(\eta) = se_m(\eta, \gamma_2^2)$

(3.23) $\quad Fek_m(\xi_0) = Fek_m(\xi_0, \gamma_2^2)$

(3.24) $\quad ce_m^*(\eta) = ce_m(\eta, \gamma_2^2)$.

Multiplying both sides of Eqs. (3.13) and (3.16) by $se_n(\eta)$ and Eqs. (3.14) and (3.15) by $ce_n(\eta)$, integrating with respect to η from 0 to 2π, and applying the orthogonality relations of the angular Mathieu functions

(3.25) $\quad \int_0^{2\pi} ce_m ce_n \, d\eta = 0 \qquad \text{if } m \neq n$

leads to

(3.26) $\quad A_{1n}Se_n = \sum_{m=1}^{\infty}{}' A_{2m} \, Gek_m \beta_{m,n}$

(3.27) $\quad B_{1n}Ce_n = \sum_{m=0}^{\infty}{}' B_{2m} \, Fek_m \alpha_{m,n}$

(3.28) $\quad \beta \sum_{m=1}^{\infty}{}' A_{1m}Se_m \nu_{n,m} - v\mu B_{1n}Ce_n' =$

$$\beta\gamma_{1h}^2/\gamma_2^2 \sum_{m=1}^{\infty}{}' A_{2m}Gek_m \, \nu_{n,m} - v\mu\gamma_{1h}^2/\gamma_2^2 \sum_{m=0}^{\infty}{}' B_{2m}Fek_m'\alpha_{n,m}$$

(3.29) $\quad v\epsilon_1 A_{1n}Se_n + \beta \sum_{m=0}^{\infty}{}' B_{1m}Ce_m\psi_{n,m} =$

$$v\epsilon_0\gamma_{1h}^2/\gamma_2^2 \sum_{m=1}^{\infty}{}' A_{2m}Gek_m'\beta_{n,m} + \beta\gamma_{1h}^2/\gamma_2^2 \sum_{m=0}^{\infty}{}' B_{2m}Fek_m\psi_{n,m}$$

The prime over the summation sign is used to indicate that either odd or even values of m are used accordingly as to whether n is odd or even.

$\alpha_{m,n}$, $\beta_{m,n}$, $\psi_{m,n}$ and $\nu_{m,n}$ are given by the following

(3.30) $\quad \alpha_{m,n} = \int_0^{2\pi} ce_m^*(\eta) \, ce_n(\eta) \, d\eta \Big/ \int_0^{2\pi} ce_n^2(\eta) \, d\eta$

(3.31) $\quad \beta_{m,n} = \int_0^{2\pi} se_m^*(\eta) \, se_n(\eta) \, d\eta \Big/ \int_0^{2\pi} se_n^2(\eta) \, d\eta$

(3.32) $\quad \psi_{m,n} = \int_0^{2\pi} ce_m'(\eta) \, se_n(\eta) \, d\eta \Big/ \int_0^{2\pi} se_n^2(\eta) \, d\eta$

(3.33) $\quad \nu_{m,n} = \int_0^{2\pi} se_m'(\eta) \, ce_n(\eta) \, d\eta \Big/ \int_0^{2\pi} ce_n^2(\eta) \, d\eta$.

Making use of Eqs. (3.26) and (3.27), Eqs. (3.28) and (3.29) yields two sets of infinite homogeneous equations

(3.34)
$$\sum_{m=1}^{\infty}{}' A_{2m}s_{m,n} + \sum_{m=0}^{\infty}{}' B_{2m}t_{m,n} = 0$$
$$\sum_{m=1}^{\infty}{}' A_{2m}g_{m,n} + \sum_{m=0}^{\infty}{}' B_{2m}h_{m,n} = 0$$

where

$$g_{m,n} = -(1 - \gamma_{1h}^2/\gamma_2^2)Gek_m(\xi_0)\sum_{r=1}^{\infty}{}' \beta_{m,r}\,\nu_{r,n} \tag{3.35}$$

$$h_{m,n} = v\mu\alpha_{m,n}/\beta\,[\,Fek_m(\xi_0)Ce_m'(\xi_0)/Ce_m(\xi_0) - Fek_m'(\xi_0)\gamma_{1h}^2/\gamma_2^2] \tag{3.36}$$

$$s_{m,n} = v\beta_{m,n}/\beta\,[\,\varepsilon_1 Gek_m(\xi_0)Se_m'(\xi_0)/Se_m(\xi_0) - \epsilon_0 Gek_m'(\xi_0)\gamma_{1h}^2/\gamma_2^2] \tag{3.37}$$

$$t_{m,n} = (1 - \gamma_{1h}^2/\gamma_2^2)Fek_m(\xi_0)\sum_{r=0}^{\infty}{}' \alpha_{m,r}\,\psi_{r,n} \tag{3.38}$$

For a nontrivial solution, the infinite determinant of Eq.(3.34) must vanish. The propagation constant β can then be determined from the roots of this infinite determinant.

The infinite determinant for odd values of m and n is

$$\begin{vmatrix} s11 & t11 & s31 & t31 & - & - \\ g11 & h11 & g31 & h31 & - & - \\ s13 & t13 & s33 & t33 & - & - \\ g13 & h13 & g33 & h33 & - & - \\ \vdots & \vdots & \vdots & \vdots & - & - \\ \vdots & \vdots & \vdots & \vdots & - & - \end{vmatrix} = 0 \tag{3.39}$$

and for even values of m and n is

$$\begin{vmatrix} h00 & g20 & h20 & g40 & - & - \\ t02 & s22 & t22 & s42 & - & - \\ h02 & g22 & h22 & g42 & - & - \\ t04 & s24 & t24 & s44 & - & - \\ \vdots & \vdots & \vdots & \vdots & - & - \\ \vdots & \vdots & \vdots & \vdots & - & - \end{vmatrix} = 0 \tag{3.40}$$

3.2 ODD MODES

The axial components of the field for odd modes are

$$E_{z1} = \sum_{m=0}^{\infty} A_{1m} Ce_m(\xi, \gamma_{1e}^2) ce_m(\eta, \gamma_{1e}^2)$$

(3.41)

$$H_{z1} = \sum_{m=1}^{\infty} B_{1m} Se_m(\xi, \gamma_{1h}^2) se_m(\eta, \gamma_{1h}^2)$$

for $0 \le \xi \le \xi_0$

$$E_{z2} = \sum_{m=0}^{\infty} A_{2m} Fek_m(\xi, \gamma_2^2) ce_m(\eta, \gamma_2^2)$$

(3.42)

$$H_{z2} = \sum_{m=1}^{\infty} B_{2m} Gek_m(\xi, \gamma_2^2) se_m(\eta, \gamma_2^2)$$

for $\xi_0 \le \xi < \infty$

where A_{im} and B_{im}, $i = 1,2$ are arbitrary constants, and γ_{1e}^2, γ_{1h}^2, and γ_2^2 are given in Eq.(3.4).

The transverse field components are

for $0 \le \xi \le \xi_0$

(3.43)
$$E_{\xi 1} = -i/(v^2\mu\epsilon_1 - \beta^2)L \left[\beta \sum_{m=0}^{\infty} A_{1m} Ce_m'(\xi, \gamma_{1e}^2) ce_m(\eta, \gamma_{1e}^2) + v\mu \sum_{m=1}^{\infty} B_{1m} Se_m(\xi, \gamma_{1h}^2) se_m'(\eta, \gamma_{1h}^2)\right]$$

(3.44)
$$E_{\eta 1} = -i/(v^2\mu\epsilon_1 - \beta^2)L \left[\beta \sum_{m=0}^{\infty} A_{1m} Ce_m(\xi, \gamma_{1e}^2) ce_m'(\eta, \gamma_{1e}^2) - v\mu \sum_{m=1}^{\infty} B_{1m} Se_m'(\xi, \gamma_{1h}^2) se_m(\eta, \gamma_{1h}^2)\right]$$

(3.45)
$$H_{\xi 1} = -i/(v^2\mu\epsilon_1 - \beta^2)L \left[-v\epsilon_1 \sum_{m=0}^{\infty} A_{1m} Ce_m(\xi, \gamma_{1e}^2) ce_m'(\eta, \gamma_{1e}^2) + \beta \sum_{m=1}^{\infty} B_{1m} Se_m'(\xi, \gamma_{1h}^2) se_m(\eta, \gamma_{1h}^2)\right]$$

(3.46)
$$H_{\eta 1} = -i/(v^2\mu\epsilon_1 - \beta^2)L \left[v\epsilon_1 \sum_{m=0}^{\infty} A_{1m} Ce_m'(\xi, \gamma_{1e}^2) ce_m(\eta, \gamma_{1e}^2) + \beta \sum_{m=1}^{\infty} B_{1m} Se_m(\xi, \gamma_{1h}^2) se_m'(\eta, \gamma_{1h}^2)\right]$$

for $\xi_0 \le \xi < \infty$

(3.47) $$E_{\xi 2} = -i/(v^2\mu\epsilon_0 - \beta^2)L \left[\beta \sum_{m=0}^{\infty} A_{2m} Fek_m'(\xi, \gamma_2^2) ce_m(\eta, \gamma_2^2) + v\mu \sum_{m=1}^{\infty} B_{2m} Gek_m(\xi, \gamma_2^2) se_m'(\eta, \gamma_2^2) \right]$$

(3.48) $$E_{\eta 2} = -i/(v^2\mu\epsilon_0 - \beta^2)L \left[\beta \sum_{m=0}^{\infty} A_{2m} Fek_m(\xi, \gamma_2^2) ce_m'(\eta, \gamma_2^2) - v\mu \sum_{m=1}^{\infty} B_{2m} Gek_m'(\xi, \gamma_2^2) se_m(\eta, \gamma_2^2) \right]$$

(3.49) $$H_{\xi 2} = -i/(v^2\mu\epsilon_0 - \beta^2)L \left[-v\epsilon_0 \sum_{m=0}^{\infty} A_{2m} Fek_m(\xi, \gamma_2^2) ce_m'(\eta, \gamma_2^2) + \beta \sum_{m=1}^{\infty} B_{2m} Gek_m'(\xi, \gamma_2^2) se_m(\eta, \gamma_2^2) \right]$$

(3.50) $$H_{\eta 2} = -i/(v^2\mu\epsilon_0 - \beta^2)L \left[v\epsilon_0 \sum_{m=0}^{\infty} A_{2m} Fek_m'(\xi, \gamma_2^2) ce_m(\eta, \gamma_2^2) + \beta \sum_{m=1}^{\infty} B_{2m} Gek_m(\xi, \gamma_2^2) se_m'(\eta, \gamma_2^2) \right]$$

The derivative with respect to ξ or η is denoted by the prime.

Equating the tangential fields at the boundary surface, $\xi = \xi_0$, gives

(3.51) $$\sum_{m=0}^{\infty} A_{1m} Ce_m(\xi_0) ce_m(\eta) = \sum_{m=0}^{\infty} A_{2m} Fek_m(\xi_0) ce_m^*(\eta)$$

(3.52) $$\sum_{m=1}^{\infty} B_{1m} Se_m(\xi_0) se_m(\eta) = \sum_{m=1}^{\infty} B_{2m} Gek_m(\xi_0) se_m^*(\eta)$$

(3.53) $$1/(v^2\mu\epsilon_1 - \beta^2) \left[\beta \sum_{m=0}^{\infty} A_{1m} Ce_m(\xi_0) ce_m'(\eta) - v\mu \sum_{m=1}^{\infty} B_{1m} Se_m'(\xi_0) se_m(\eta) \right]$$
$$= 1/(v^2\mu\epsilon_0 - \beta^2) \left[\beta \sum_{m=0}^{\infty} A_{2m} Fek_m(\xi_0) ce_m^{*\prime}(\eta) - v\mu \sum_{m=1}^{\infty} B_{2m} Gek_m'(\xi_0) se_m^*(\eta) \right]$$

(3.54) $$1/(v^2\mu\epsilon_1 - \beta^2) \left[v\epsilon_1 \sum_{m=0}^{\infty} A_{1m} Ce_m'(\xi_0) ce_m(\eta) + \beta \sum_{m=1}^{\infty} B_{1m} Se_m(\xi_0) se_m'(\eta) \right]$$
$$= 1/(v^2\mu\epsilon_0 - \beta^2) \left[v\epsilon_0 \sum_{m=0}^{\infty} A_{2m} Fek_m'(\xi_0) ce_m^*(\eta) + \beta \sum_{m=1}^{\infty} B_{2m} Gek_m(\xi_0) se_m^{*\prime}(\eta) \right]$$

The abbreviations

(3.55) $\quad Ce_m(\xi_0) = Ce_m(\xi_0, \gamma_{1e}^2)$

(3.56) $\quad ce_m(\eta) = ce_m(\eta, \gamma_{1e}^2)$

(3.57) $\quad Se_m(\xi_0) = Se_m(\xi_0, \gamma_{1h}^2)$

(3.58) $\quad se_m(\eta) = se_m(\eta, \gamma_{1h}^2)$

(3.59) $\quad Fek_m(\xi_0) = Fek_m(\xi_0, \gamma_2^2)$

(3.60) $\quad ce_m^*(\eta) = ce_m(\eta, \gamma_2^2)$

(3.61) $\quad Gek_m(\xi_0) = Gek_m(\xi_0, \gamma_2^2)$

(3.62) $\quad se_m^*(\eta) = se_m(\eta, \gamma_2^2)$

have been used.

Multiplying both sides of Eqs. (3.51) and (3.54) by $ce_n(\eta)$ and Eqs. (3.52) and (3.53) by $se_n(\eta)$, integrating with respect to η from 0 to 2π, and applying the orthogonality relations of the angular Mathieu functions leads to

(3.63) $\quad A_{1n}Ce_n = \sum_{m=0}^{\infty}{}' A_{2m}Fek_m\beta_{m,n}$

(3.64) $\quad B_{1n}Se_n = \sum_{m=1}^{\infty}{}' B_{2m}Gek_m\alpha_{m,n}$

(3.65) $\quad \beta \sum_{m=0}^{\infty}{}' A_{1m}Ce_m\nu_{n,m} - v\mu B_{1n}Se_n' =$

$$\beta\gamma_{1h}^2/\gamma_2^2 \sum_{m=0}^{\infty}{}' A_{2m}Fek_m \nu_{n,m} - v\mu\gamma_{1h}^2/\gamma_2^2 \sum_{m=1}^{\infty}{}' B_{2m}Gek_m'\alpha_{n,m}$$

(3.66) $\quad v\epsilon_1 A_{1n}Ce_n + \beta \sum_{m=1}^{\infty}{}' B_{1m}Se_m\psi_{n,m} =$

$$v\epsilon_0\gamma_{1h}^2/\gamma_2^2 \sum_{m=0}^{\infty}{}' A_{2m}Fek_m'\beta_{n,m} + \beta\gamma_{1h}^2/\gamma_2^2 \sum_{m=1}^{\infty}{}' B_{2m}Gek_m\psi_{n,m}$$

The prime over the summation sign is used to indicate that either odd or even values of m are used accordingly as to whether n is odd or even.

$\alpha_{m,n}$, $\beta_{m,n}$, $\psi_{m,n}$ and $\nu_{m,n}$ are given by the following

(3.67) $\quad \alpha_{m,n} = \int_0^{2\pi} se_m^*(\eta)se_n(\eta)\, d\eta \Big/ \int_0^{2\pi} se_n^2(\eta)\, d\eta$

(3.68) $\quad \beta_{m,n} = \int_0^{2\pi} ce_m^*(\eta)ce_n(\eta)\, d\eta \Big/ \int_0^{2\pi} ce_n^2(\eta)\, d\eta$

(3.69) $\quad \psi_{m,n} = \int_0^{2\pi} se_m'(\eta)ce_n(\eta)\, d\eta \Big/ \int_0^{2\pi} ce_n^2(\eta)\, d\eta$

(3.70) $\quad \nu_{m,n} = \int_0^{2\pi} ce_m'(\eta)se_n(\eta)\, d\eta \Big/ \int_0^{2\pi} se_n^2(\eta)\, d\eta$

Making use of Eqs. (3.63) and (3.64), Eqs. (3.65) and (3.66) yields two sets of infinite homogeneous equations

$$(3.71)\qquad \begin{aligned} \sum_{m=0}^{\infty}{}' A_{2m} s_{m,n} + \sum_{m=1}^{\infty}{}' B_{2m} t_{m,n} = 0 \\ \sum_{m=0}^{\infty}{}' A_{2m} g_{m,n} + \sum_{m=1}^{\infty}{}' B_{2m} h_{m,n} = 0 \end{aligned}$$

where

$$(3.72)\qquad g_{m,n} = -(1 - \gamma_{1h}^2/\gamma_2^2) Fek_m(\xi_0) \sum_{r=0}^{\infty}{}' \beta_{m,r} \nu_{r,n}$$

$$(3.73)\qquad h_{m,n} = v\mu\alpha_{m,n}/\beta \; [\, Gek_m(\xi_0) Se_m'(\xi_0)/Se_m(\xi_0) - Gek_m'(\xi_0)\, \gamma_{1h}^2/\gamma_2^2 \,]$$

$$(3.74)\qquad s_{m,n} = v\beta_{m,n}/\beta \; [\, \epsilon_1 Fek_m(\xi_0) Ce_m'(\xi_0)/Ce_m(\xi_0) - \epsilon_0 Fek_m'(\xi_0)\, \gamma_{1h}^2/\gamma_2^2 \,]$$

$$(3.75)\qquad t_{m,n} = (1 - \gamma_{1h}^2/\gamma_2^2) Gek_m(\xi_0) \sum_{r=1}^{\infty}{}' \alpha_{m,r} \Psi_{r,n}$$

For a nontrivial solution, the infinite determinant of Eq. (3.71) must vanish. The propagation constant β can then be determined from the roots of this infinite determinant.

The infinite detеminant for odd values of m and n is

$$(3.76)\qquad \begin{vmatrix} s11 & t11 & s31 & t31 & - & - \\ g11 & h11 & g31 & h31 & - & - \\ s13 & t13 & s33 & t33 & - & - \\ g13 & h13 & g33 & h33 & - & - \\ : & : & : & : & - & - \\ : & : & : & : & - & - \end{vmatrix} = 0$$

and for even values of m and n is

$$
(3.77)\qquad \begin{vmatrix} s00 & s20 & t20 & s40 & - & - \\ s02 & s22 & t22 & s42 & - & - \\ g02 & g22 & h22 & g42 & - & - \\ s04 & s24 & t24 & s44 & - & - \\ \vdots & \vdots & \vdots & \vdots & - & - \\ \vdots & \vdots & \vdots & \vdots & - & - \end{vmatrix} = 0
$$

4. WEAKLY GUIDING APPROXIMATION

The exact characteristic equations obtained in Chapter 3 are valid for an anisotropic elliptical fiber with any eccentricities. These equations are also applicable to the fiber with any refractive index differences between the core and cladding material. However, for most of practical fibers, the difference in the refractive indices of the core and the cladding is typically very small. The simplified characteristic equations can be obtained under this condition which is known as the weakly guiding approximation.

Applying the weakly guiding approximation results in the following

(4.1) $\qquad \gamma_2^2 = \gamma_{1h}^2 + \not{k}v^2\mu\varepsilon_1\,(1 - \varepsilon_0/\varepsilon_1) \approx \gamma_{1h}^2$

(4.2) $\qquad 1 - \gamma_{1h}^2/\gamma_2^2 \approx 0.$

4.1 EVEN MODES

Applying Eqs.(4.1) and (4.2) into Eqs.(3.20) and (3.30) yields the equations

(4.3) $\qquad ce_n(\eta\,,\, \gamma_2^2) \approx ce_n(\eta\,,\, \gamma_{1h}^2)$

(4.4) $\qquad \alpha_{m,n} \approx \int_0^{2\pi} ce_m(\eta\,,\, \gamma_{1h}^2)ce_n(\eta\,,\, \gamma_{1h}^2)d\eta \Big/ \int_0^{2\pi} ce_m^2(\eta\,,\, \gamma_{1h}^2)d\eta$

$\qquad\qquad = \Delta_{m,n}$

where $\Delta_{m,n}$ is the Kronecker delta which is zero when $m \neq n$ and is unity when $m = n$.

Substituting Equations (4.1) - (4.4) into Equations (3.35) - (3.38), the infinite determinants for even modes become

(4.5) $\qquad \prod_m (\, g_{m,m}\, t_{m,m} - h_{m,m}\, s_{m,m}) = 0$

or

(4.6) $\qquad g_{m,m}t_{m,m} - h_{m,m}s_{m,m} = 0$

for $m = 0, 1, 2, - - -$.

By substituting Equations (3.35) - (3.38) into Eq.(4.6), the following equation is obtained

$$-(1 - \gamma_{1h}^2/\gamma_2^2)(\epsilon_1/\epsilon_0 - \gamma_{1h}^2/\gamma_2^2)(\sum_{n=1}^{\infty}{}' \beta_{m,n}\, \nu_{n,m} \sum_{n=0}^{\infty}{}' \alpha_{m,n}\, \Psi_{n,m}/\alpha_{m,m}\beta_{m,m})$$

$$= [Ce_m'(\xi_0)/Ce_m(\xi_0) - (\gamma_{1h}^2/\gamma_2^2)Fek_m'(\xi_0)/Fek_m(\xi_0)] \quad (4.7)$$

$$[(\epsilon_1/\epsilon_0)Se_m'(\xi_0)/Se_m(\xi_0) - (\gamma_{1h}^2/\gamma_2^2)Gek_m'(\xi_0)/Gek_m(\xi_0)].$$

This is the simplified characteristic equation for even modes compared to the infinite determinants as given in Eqs.(3.39) and (3.40). When the elliptical rod degenerates to a circular rod, the simplified characteristic equation becomes that of the anisotropic circular fiber.

4.2 ODD MODES

Applying Eqs.(4.1) and (4.2) in Eqs.(3.58) and (3.67) yields the equations

$$se_n(\eta, \gamma_2^2) = se_n(\eta, \gamma_{1h}^2) \quad (4.8)$$

$$\alpha_{m,n} = \int_0^{2\pi} se_m(\eta, \gamma_{1h}^2) se_n(\eta, \gamma_{1h}^2) d\eta / \int_0^{2\pi} se_m^2(\eta, \gamma_{1h}^2) d\eta \quad (4.9)$$

$$= \Delta_{m,n}$$

where $\Delta_{m,n}$ is the Kronecker delta which is zero when $m \neq n$ and is unity when $m = n$.

Substituting Equations (4.1) - (4.2) and (4.8) - (4.9) into Equations (3.72) - (3.75), the infinite determinants for odd modes become

$$\prod_m (g_{m,m}\, t_{m,m} - h_{m,m}\, s_{m,m}) = 0 \quad (4.10)$$

or

$$g_{m,m} t_{m,m} - h_{m,m} s_{m,m} = 0 \quad (4.11)$$

for $m = 0, 1, 2, ---$.

By substituting Equations (3.72) - (3.75) into Eq.(4.11), the following equation is obtained

$$-(1 - \gamma_{1h}^2/\gamma_2^2)(\epsilon_1/\epsilon_0 - \gamma_{1h}^2/\gamma_2^2)(\sum_{n=0}^{\infty}{}'\beta_{m,n}\,\nu_{n,m}\sum_{n=1}^{\infty}{}'\alpha_{m,n}\,\Psi_{n,m}/\alpha_{m,m}\beta_{m,m})$$

$$= [Se_m'(\xi_0)/Se_m(\xi_0) - (\gamma_{1h}^2/\gamma_2^2)Gek_m'(\xi_0)/Gek_m(\xi_0)] \tag{4.12}$$

$$[(\epsilon_1/\epsilon_0)Ce_m'(\xi_0)/Ce_m(\xi_0) - (\gamma_{1h}^2/\gamma_2^2)Fek_m'(\xi_0)/Fek_m(\xi_0)].$$

This is the simplified characteristic equation for odd modes compared to the infinite determinants given in Eqs.(3.76) and (3.77). When the elliptical rod degenerates to a circular rod, the simplified characteristic equation becomes that of the anisotropic circular fiber.

5. NUMERICAL RESULTS FOR PROPAGATION CONSTANTS

5.1 ISOTROPIC ELLIPTICAL FIBERS

When the anisotropic parameter g in Eq.(3.1) is equal to unity, the simplified characteristic equations in Eq.(4.7) and Eq.(4.12) become that of an isotropic elliptic guide. In Figures 2 through 5, the normalized guide wavelength λ/λ_0 for the isotropic elliptical fibers is plotted as a function of the normalized cross-section area and normalized major axis for $\epsilon_1/\epsilon_0 = 2.5$ and for the various values of ξ_0. These results are compared with those given by Yeh[2] which are indicated by symbols; the results are in close agreement.

For the eHE_{11} mode, it can be seen in Figure 2 that the normalized guide wavelength is almost equal to unity for the very small value of the cross-section area. This indicates that the geometry of the waveguide has no effect on the normalized guide wavelength when the wavelength is much larger than the physical dimension of the core of fibers. For a fixed value of cross-section area, the normalized guide wavelength is smaller for larger the value of ξ_0. This indicates that more energy is carried inside of the circular core than the elliptical core. As the normalized cross-section area becomes larger, the difference in the normalized guide wavelengths for varying ξ_0 becomes small again. This is since most of the energy is carried inside of the core and the geometry of waveguide has no effect on the normalized guide wavelength.

However, as observed in Figure 3, the oHE_{11} mode is different from the eHE_{11} mode in that the difference in the normalized guide wavelengths for varying ξ_0 is smaller than that of eHE_{11} when the value of normalized cross-section area is fixed. This small difference is due to the fact that the electric lines are being compressed such that the field density is more

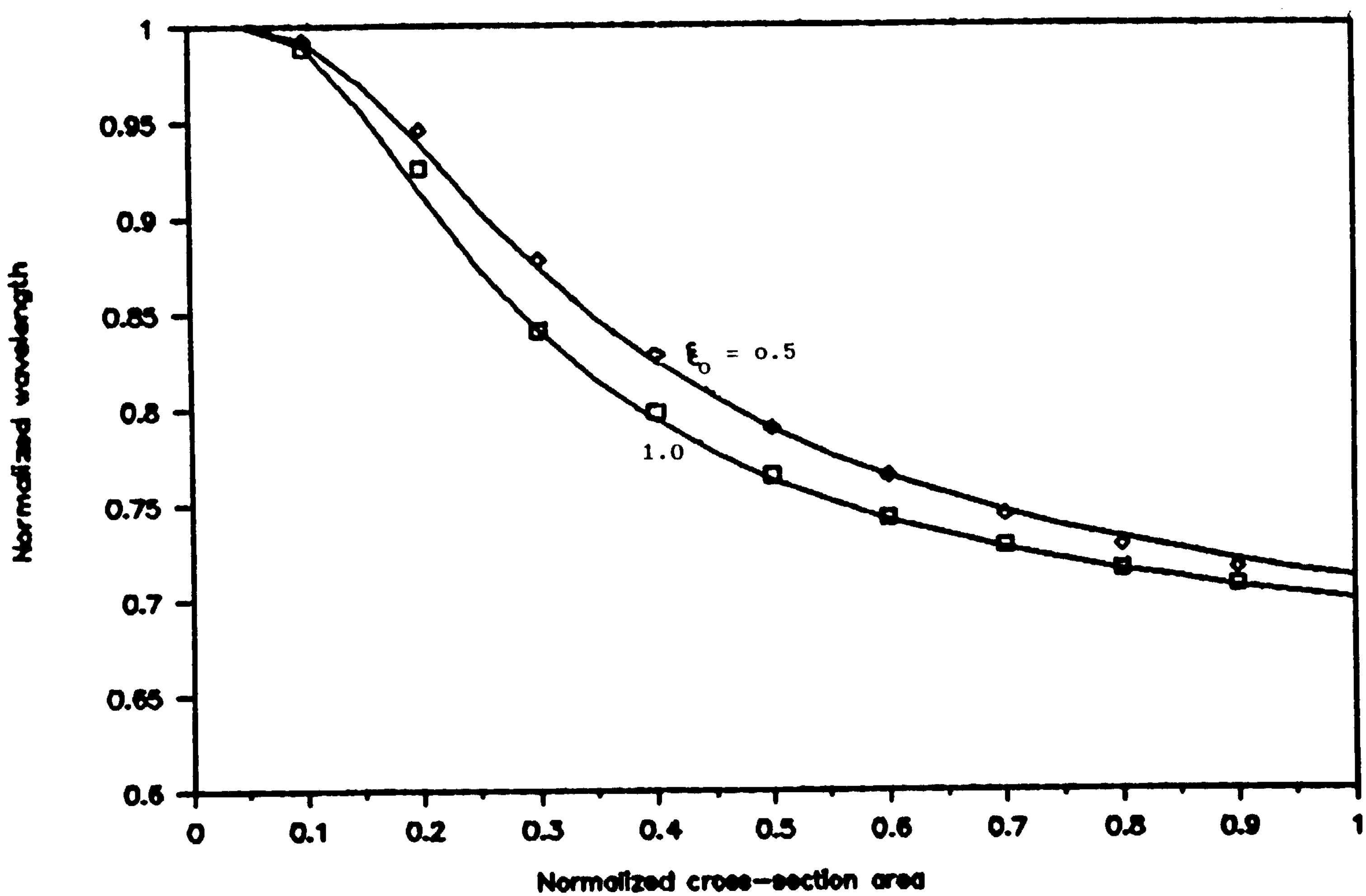


Figure 2. Normalized wavelength for isotropic fiber as a function of normalized cross-section area for even modes. Symbols are from Yeh[2].

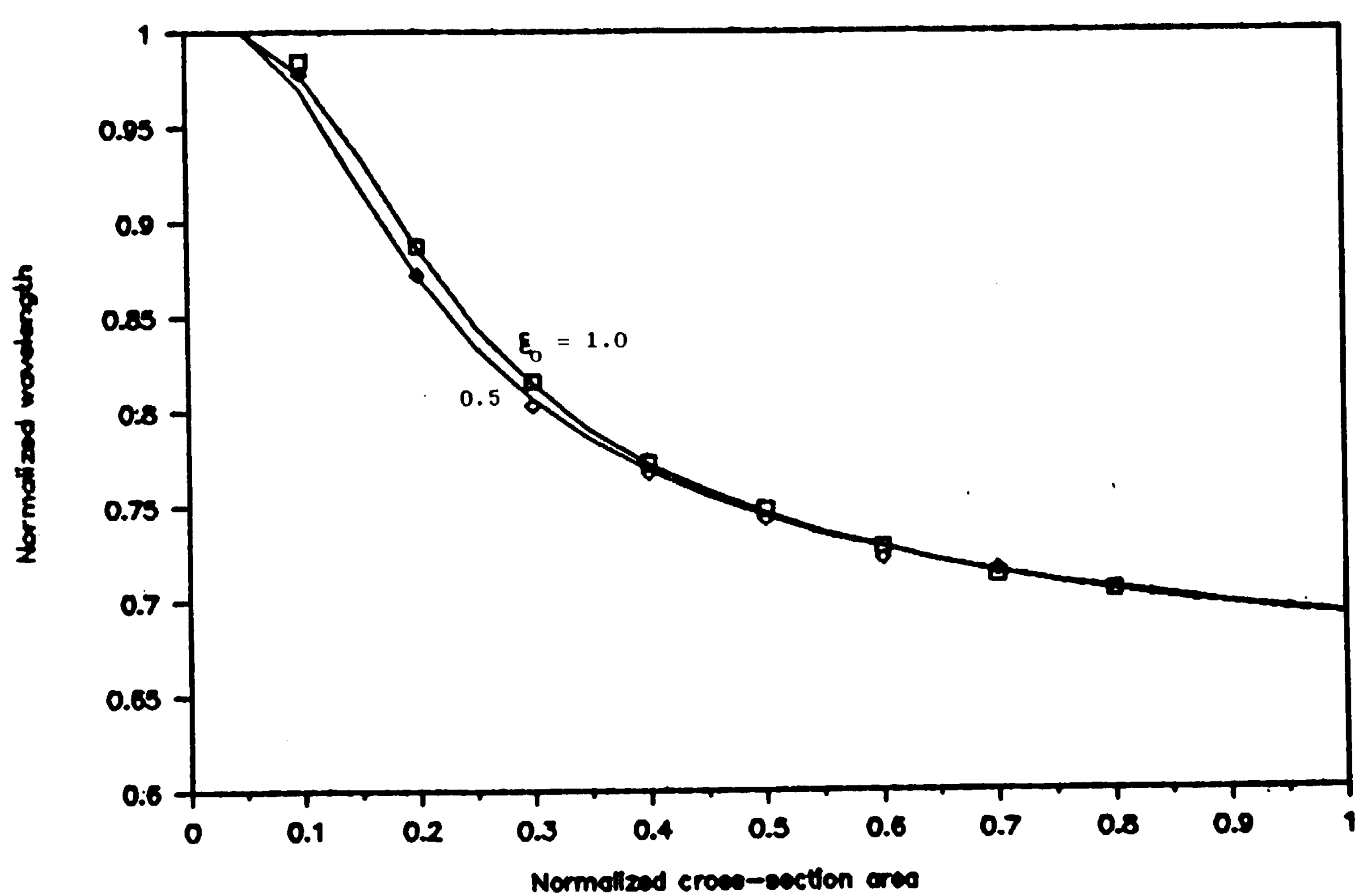


Figure 3. Normalized wavelength for isotropic fiber as a function of normalized cross-section area for odd modes. Symbols are from Yeh[2].

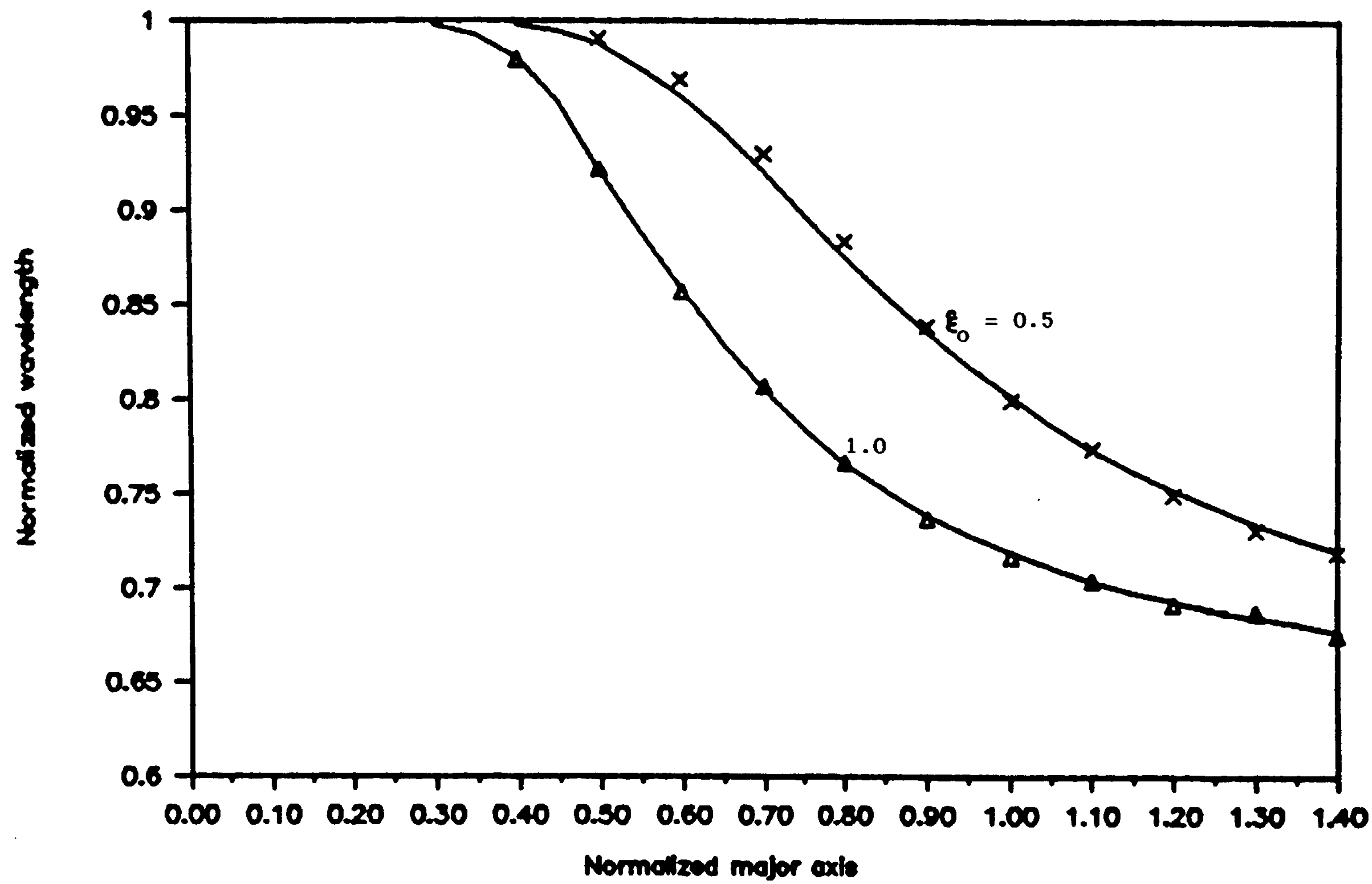


Figure 4. Normalized wavelength for isotropic fiber as a function of normalized major axis for even modes. Symbols are from Yeh[2].

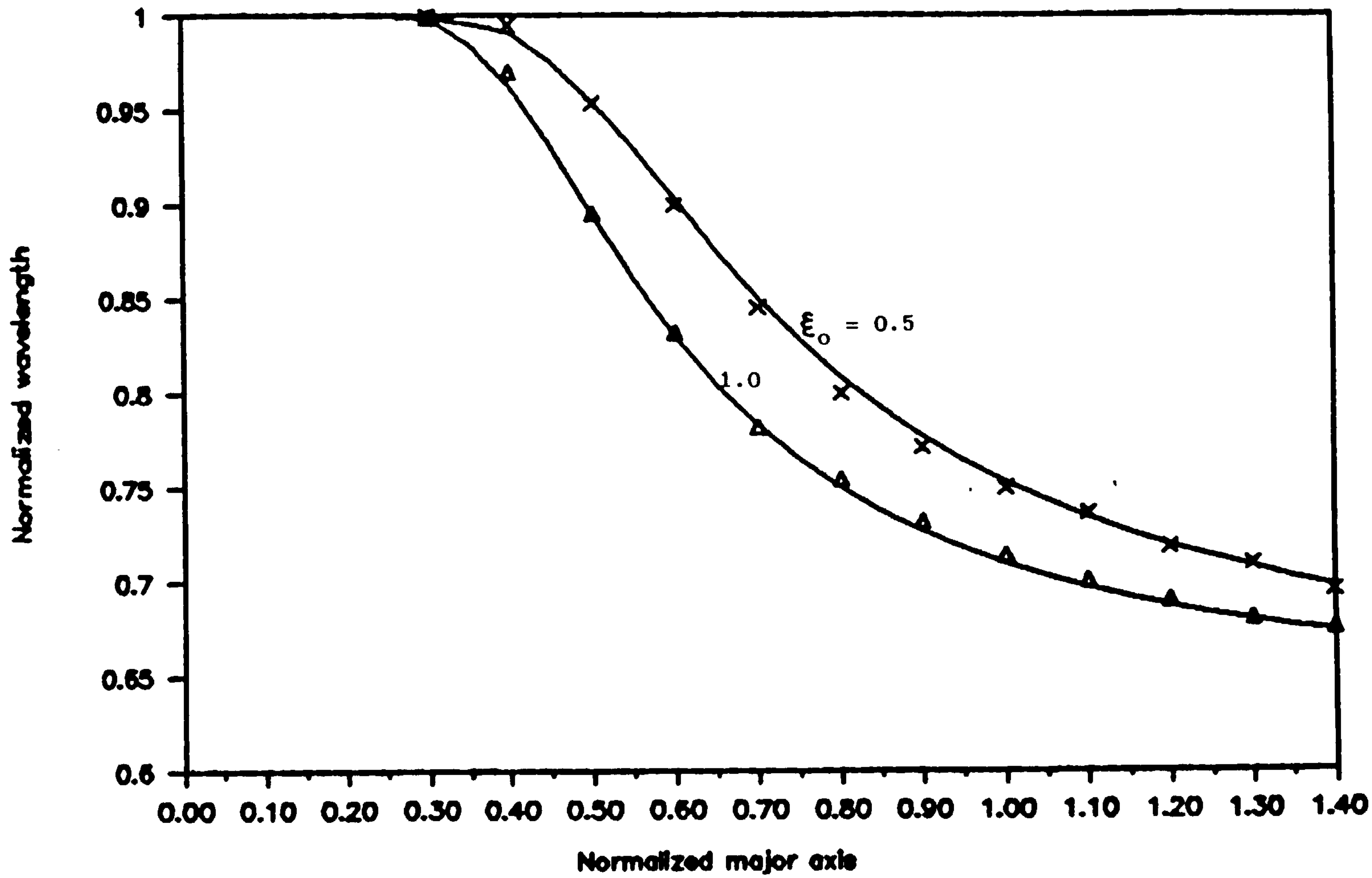


Figure 5. Normalized wavelength for isotropic fiber as a function of normalized major axis for odd modes. Symbols are from Yeh[2].

concentrated inside the waveguide. For a fixed value of cross-section area, more energy is carried inside of the elliptical core than the circular core since the normalized guide wavelength is smaller for smaller the value of ξ_0.

In Figures 4 and 5, the normalized guide wavelength is plotted against the nomalized major axis for various values of ξ_0 and for $\epsilon_1/\epsilon_0 = 2.5$. In these figures, the difference in the normalized guide wavelengths for varying ξ_0 is larger than those in Figures 2 and 3 since there is more binding dielectric material in a circular core than in a flatter rod (i.e. smaller ξ_0) when the value of normalized major axis is fixed.

5.2 ANISOTROPIC ELLIPTICAL FIBERS

In Figures 6 and 7, the normalized guide wavelength λ/λ_0 for an anisotropic elliptical fiber is plotted as a function of the normalized cross-section area for various values of anisotropy and for $\epsilon_1/\epsilon_0 = 2.5$ and $\xi_0 = 0.5$. These figures indicate that the geometry of the waveguide and anisotropy of the core have no effect when the wavelength is much larger than the physical dimention of the core of fibers which indicate that most of the energy is carried outside of the core. For a fixed value of cross-section area, the normalized guide wavelength is smaller for larger the value of anisotropy. This condition indicates that the field intensity is more concentrated in the core, thus indicating that more energy is carried inside of the core. As the normalized cross-section area becomes larger, the difference in the normalized guide wavelengths for the varying anisotropy becomes smaller again. This indicates that the geometry and anisotropy of waveguide have a smaller effect on the normalized guide wavelength.

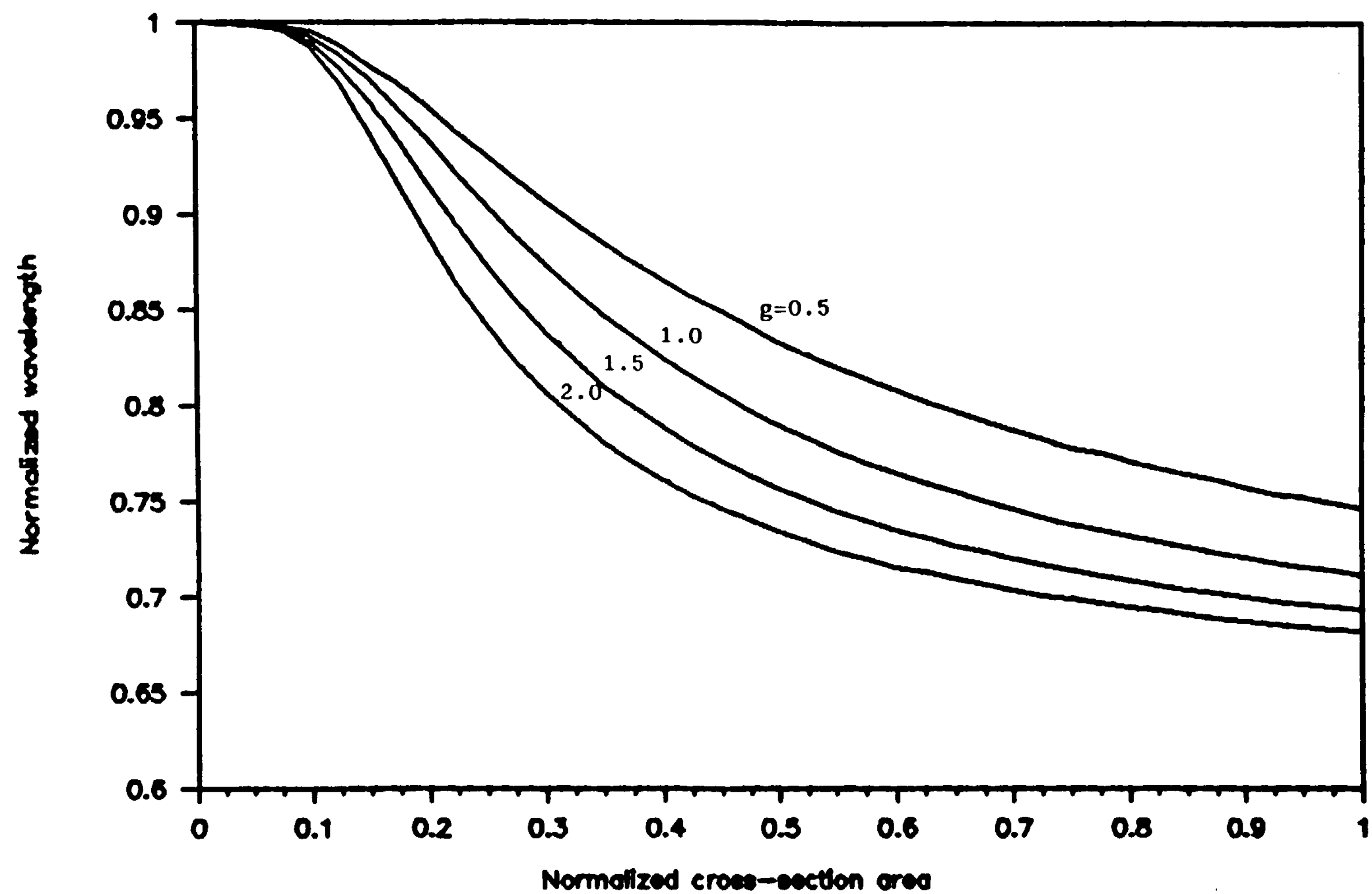


Figure 6. Normalized wavelength for anisotropic fiber as a function of normalized cross-section area for even modes.

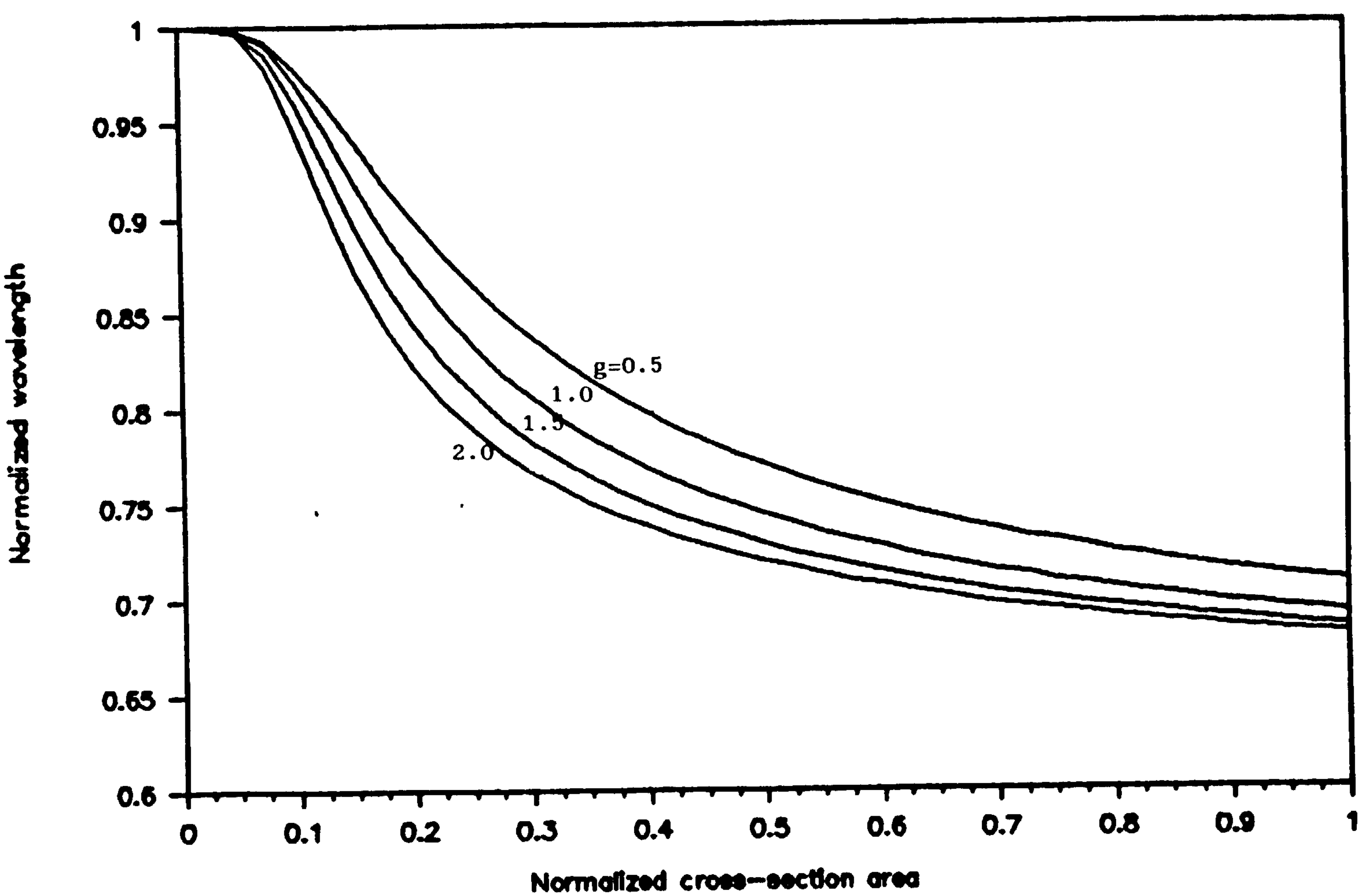


Figure 7. Normalized wavelength for anisotropic fiber as a function of normalized cross-section area for odd modes.

The nomalized guide wavelength in Figures 8 and 9 is plotted against the nomalized major axis for the various values of anisotropy and for ϵ_1/ϵ_0 = 2.5 and ξ_0 = 0.5. The effect of anisotropy on the normalized guide wavelength is similar to those in Figures 6 and 7.

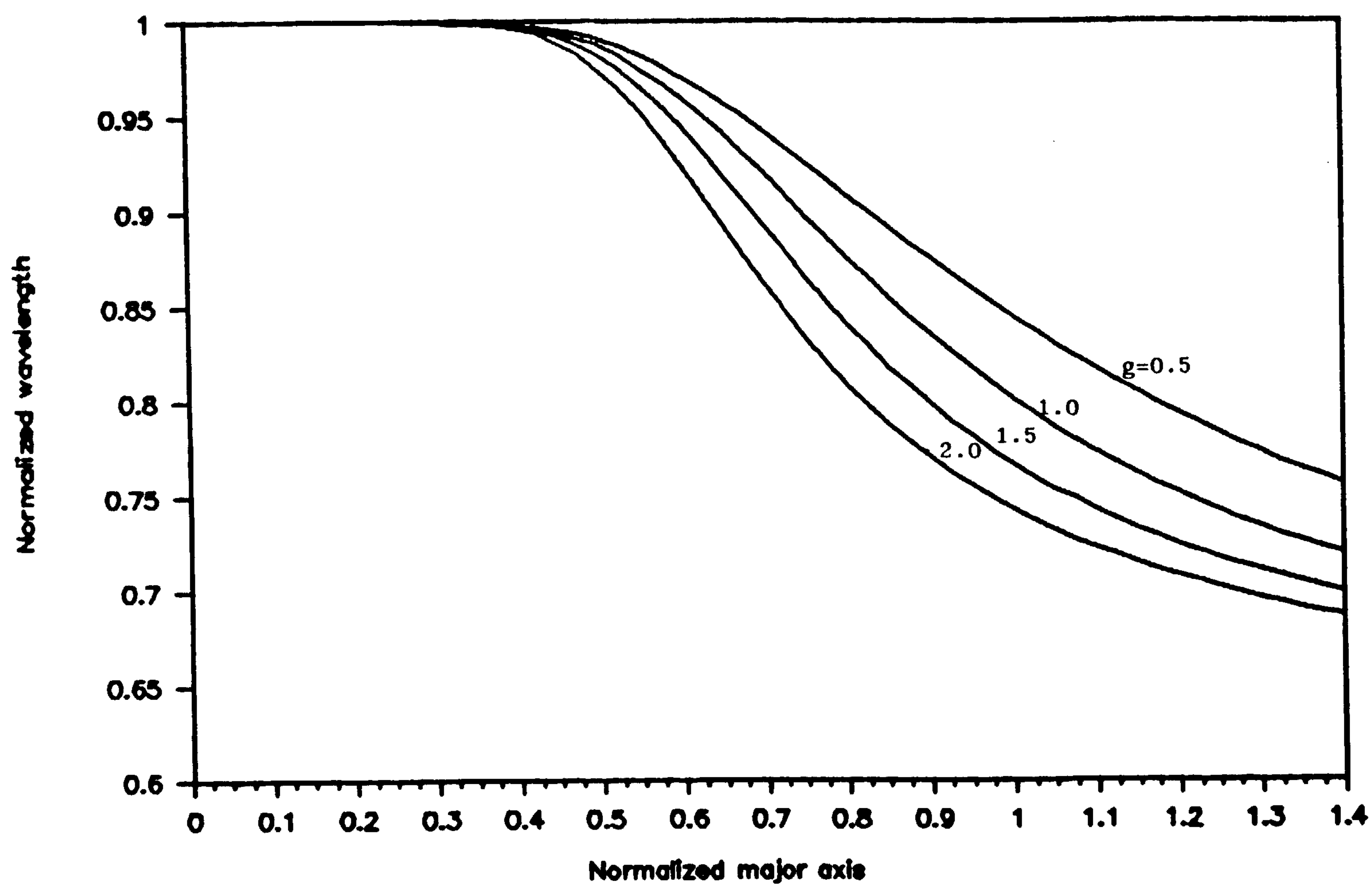


Figure 8. Normalized wavelength for anisotropic fiber as a function of normalized major axis for even modes.

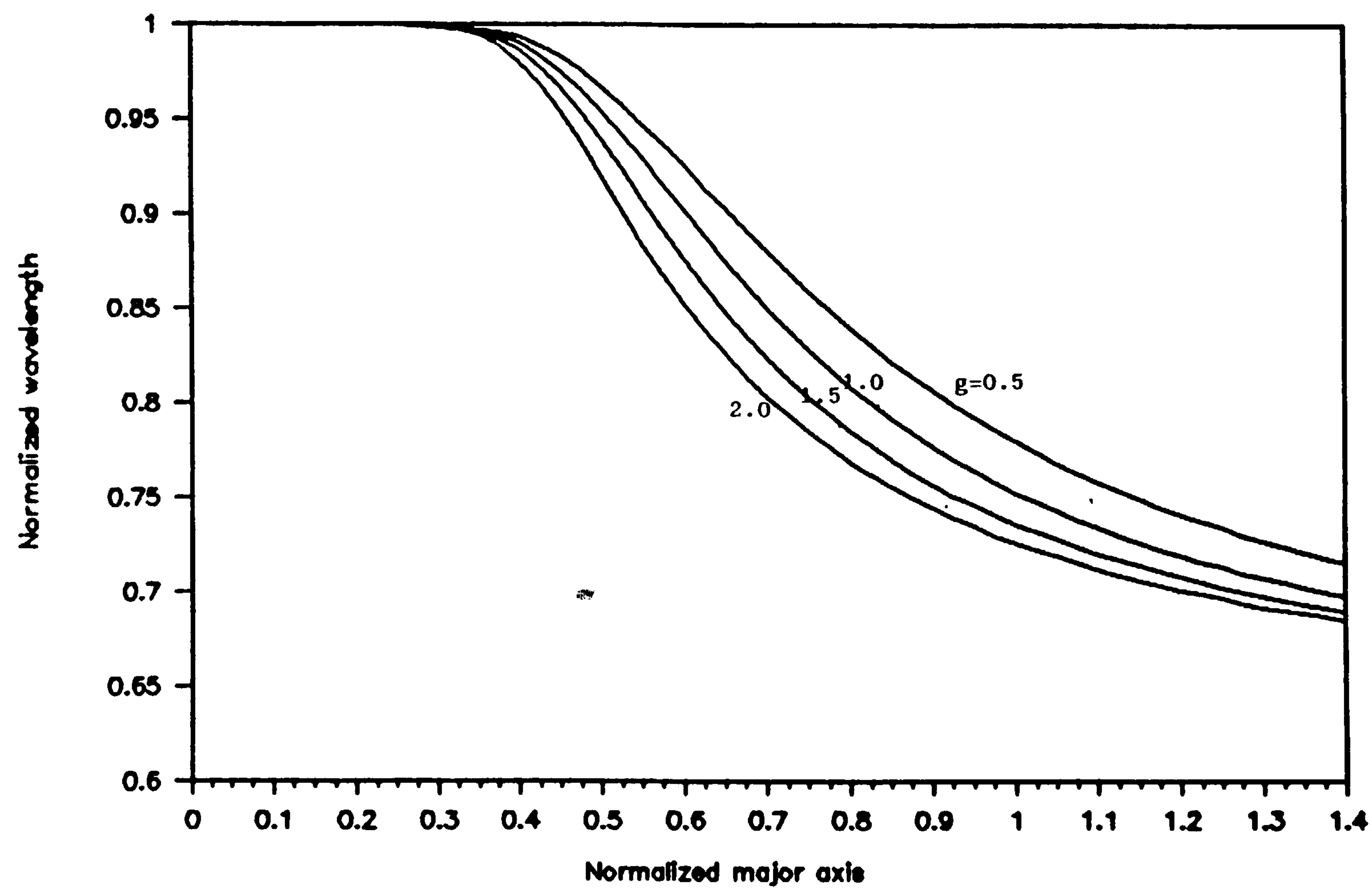


Figure 9. Normalized wavelength for anisotropic fiber as a function of normalized major axis for odd modes.

6. POWER CONSIDERATIONS

The power along the z axis in the medium i of the fiber may be obtained by integrating the poynting vector over the surface area,

$$
(6.1) \qquad
\begin{aligned}
P_i &= 1/2 \int_S (\bar{E}_t \times \bar{H}_t^*) \cdot \hat{z}\, ds \\
&= 1/2 \int_{\xi_{i-1}}^{\xi_i} \int_0^{2\pi} (E_{\xi i} H_{\eta i}^* - E_{\eta i} H_{\xi i}^*)\, L^2\, d\eta\, d\xi
\end{aligned}
$$

where

$$L = q[(\cosh 2\xi - \cos 2\eta)/2]^{1/2}$$

and $\xi_0 = 0$ and $\xi_2 = \infty$.

6.1 EVEN MODES

Substituting Equations (3.5) through (3.8) into Eq.(6.1) and integrating over the core area yields

$$
\begin{aligned}
(6.2) \quad Pcore = {} & (1/2\gamma_1^4) \int_0^{\xi_0} [\beta v \varepsilon_1 \sum_{m=1}^{\infty} A_{1m}^2 \{ \pi Se_m'^2 + S_{mm} Se_m^2 \} \\
& \qquad + \beta v \mu \sum_{m=0}^{\infty} B_{1m}^2 \{ \pi Ce_m'^2 + C_{mm} Ce_m^2 \}]\, d\xi \\
& + (\beta^2 + v^2 \mu \varepsilon_1)/2\gamma_1^4 \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} B_{1m} A_{1n} T_{mn} [Ce_m Se_n]_0^{\xi_0} \\
& + \beta v \mu/2\gamma_1^4 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty}{}' B_{1m} B_{1n} C_{mn} [Ce_m' Ce_n - Ce_m Ce_n']_0^{\xi_0} (a_m - a_n)^{-1} \\
& + \beta v \varepsilon_1/2\gamma_1^4 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty}{}' A_{1m} A_{1n} S_{mn} [Se_m' Se_n - Se_m Se_n']_0^{\xi_0} (b_m - b_n)^{-1}
\end{aligned}
$$

Similary, the power carried in the cladding is

$$
\begin{aligned}
(6.3) \quad Pclad = {} & (1/2\gamma_2^4) \int_{\xi_0}^{\infty} [\beta v \varepsilon_0 \sum_{m=1}^{\infty} A_{2m}^2 \{ \pi Gek_m'^2 + S_{mn}^* Gek_m^2 \} \\
& \qquad + \beta v \mu \sum_{m=0}^{\infty} B_{2m}^2 \{ \pi Fek_m'^2 + C_{mn}^* Fek_m^2 \}]\, d\xi \\
& + (\beta^2 + v^2 \mu \varepsilon_0)/2\gamma_2^4 \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} B_{2m} A_{2n} T_{mn}^* [Fek_m Gek_n]_{\xi_0}^{\infty} \\
& + \beta v \mu/2\gamma_2^4 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty}{}' B_{2m} B_{2n} C_{mn}^* [Fek_m' Fek_n \\
& \qquad\qquad - Fek_m Fek_n']_{\xi_0}^{\infty} (a_m - a_n)^{-1} \\
& + \beta v \varepsilon_0/2\gamma_2^4 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty}{}' A_{2m} A_{2n} S_{mn}^* [Gek_m' Gek_n \\
& \qquad\qquad - Gek_m Gek_n']_{\xi_0}^{\infty} (b_m - b_n)^{-1}
\end{aligned}
$$

In Eqs.(6.2) and (6.3), a_m and b_m are the characteristic values of the even and odd Mathieu functions of order m, respectively. The prime over the summation sign is used to indicate that m = n is excluded. Also, $\gamma_1^2 = 4\gamma_{1h}^2/q^2$ and $\gamma_2^2 = 4\gamma_2^2/q^2$ are used.

The following abbreviations have been used,

(6.4) $$C_{mn} = \int_0^{2\pi} ce_m'(\eta) ce_n'(\eta)\, d\eta$$

(6.5) $$S_{mn} = \int_0^{2\pi} se_m'(\eta) se_n'(\eta)\, d\eta$$

(6.6) $$T_{mn} = \int_0^{2\pi} ce_m'(\eta) se_n(\eta)\, d\eta$$

(6.7) $$C_{mn}^* = \int_0^{2\pi} ce_m^{*\prime}(\eta) ce_n^{*\prime}(\eta)\, d\eta$$

(6.8) $$S_{mn}^* = \int_0^{2\pi} se_m^{*\prime}(\eta) se_n^{*\prime}(\eta)\, d\eta$$

(6.9) $$T_{mn}^* = \int_0^{2\pi} ce_m^{*\prime}(\eta) se_n^{*}(\eta)\, d\eta\ .$$

The power distribution characteristics for the eHE_{11} mode is given in Figure 10. The fractional power carried by the core and cladding is plotted against the normalized major axis for the various values of anisotropy and for $\epsilon_1/\epsilon_0 = 2.5$ and $\xi_0 = 1.0$. Most of the power is carried in the cladding near the cut-off and in the core far from the cut-off. For a fixed value of the normalized major axis, the more energy is concentrated inside of the core for larger the value of anisotropy and far from the cut-off.

6.2 ODD MODES

Substituting Equations (3.43) through (3.46) into Eq.(6.1) and integrating over the core area yields

(6.10) $$\begin{aligned} \text{Pcore} = {} & (1/2\gamma_1^4)\int_0^{\xi_0} [\ \beta v \varepsilon_1 \Sigma A_{1m}^2 (\ \pi Ce_m'^2 + C_{mm} Ce_m^2\ ) \\ & \qquad + \beta v \mu \Sigma B_{1m}^2 \{\ \pi Se_m'^2 + S_{mm} Se_m^2\ \}]\, d\xi \\ & + (\ \beta^2 + v^2 \mu \varepsilon_1)/2\gamma_1^4 \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} B_{1m} A_{1n} T_{mn} [\ Se_m Ce_n]_0^{\xi_0} \\ & + \beta v \mu / 2\gamma_1^4 \sum_{m=0}^{\infty} {\sum_{n=0}^{\infty}}' B_{1m} B_{1n} S_{mn} [Se_m' Se_n - Se_m Se_n']_0^{\xi_0} (b_m - b_n)^{-1} \\ & + \beta v \varepsilon_1 / 2\gamma_1^4 \sum_{m=1}^{\infty} {\sum_{n=1}^{\infty}}' A_{1m} A_{1n} C_{mn} [Ce_m' Ce_n - Ce_m Ce_n']_0^{\xi_0} (a_m - a_n)^{-1} \end{aligned}$$

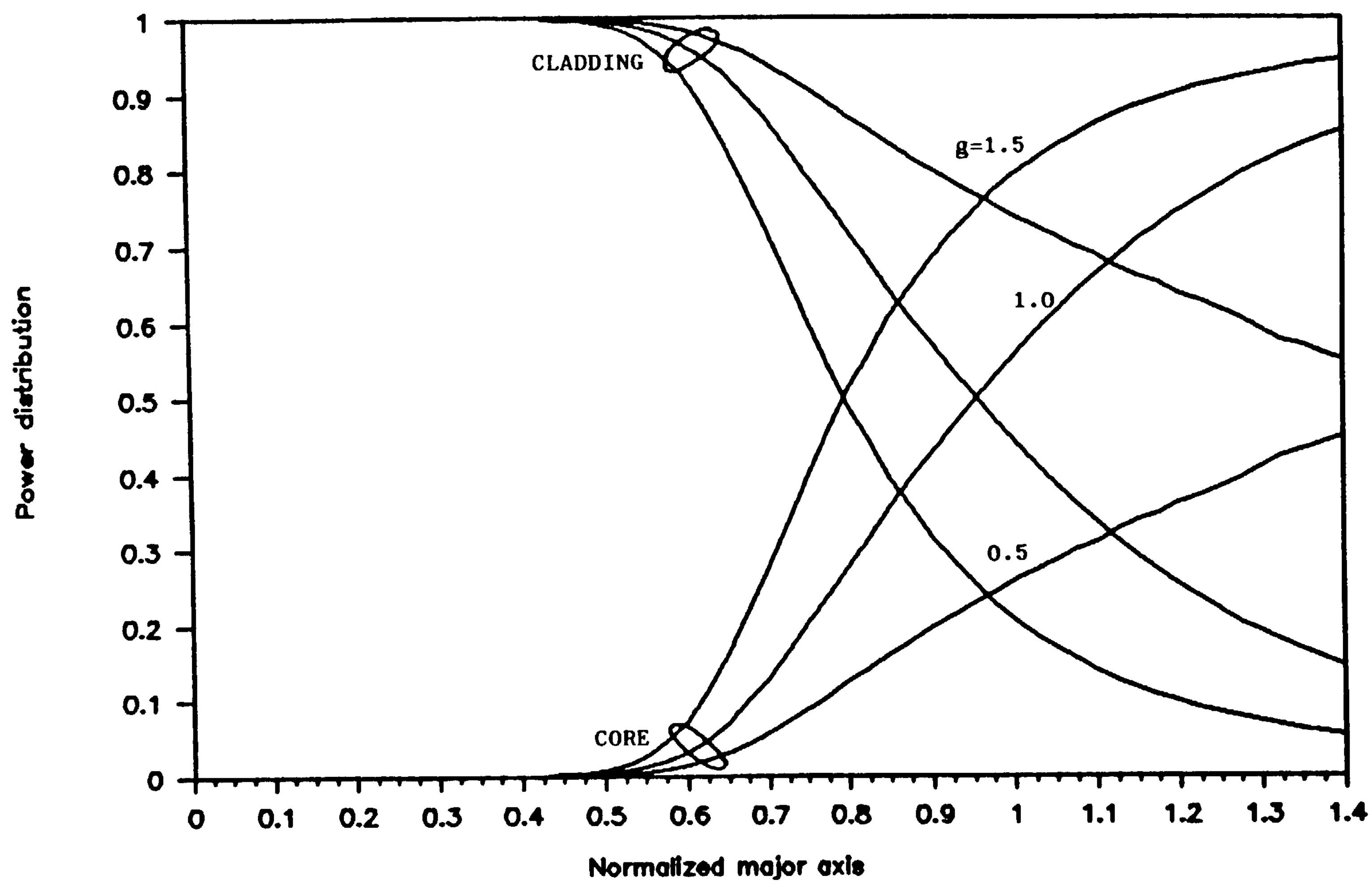


Figure 10. Power distribution characteristics for elliptical fiber as a function of normalized major axis for even modes.

Similary, the power carried in the cladding is

(6.11) $$\mathrm{Pclad} = (1/2\gamma_2^4)\int_{\xi_0}^{\infty}[\ \beta v\varepsilon_0 \sum_{m=0}^{\infty} A_{2m}^2(\ \pi Fek_m'^2 + C_{mm}^* Fek_m^2\ )$$
$$+ \beta v\mu\ \Sigma\ B_{2m}^2\ (\ \pi Gek_m'^2 + S_{mm}^* Gek_m^2\ )]\ d\xi$$
$$+ (\ \beta^2+v^2\mu\varepsilon_0)/2\gamma_2^4 \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} B_{2m}A_{2n}T_{mn}^*[\ Gek_m Fek_n]_{\xi_0}^{\infty}$$
$$+ \beta v\mu/2\gamma_2^4 \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}{}'\ B_{2m}B_{2n}S_{mn}^*[Gek_m'Gek_n$$
$$- Gek_m Gek_n']_{\xi_0}^{\infty}(b_m - b_n)^{-1}$$
$$+ \beta v\varepsilon_0/2\gamma_2^4 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}{}'\ A_{2m}A_{2n}C_{mn}^*[Fek_m'Fek_n$$
$$- Fek_m Fek_n']_{\xi_0}^{\infty}(a_m - a_n)^{-1}$$

In Eqs.(6.10) and (6.11), a_m and b_m are the characteristic values of the even and odd Mathieu functions of order m, respectively. The prime over the summation sign is used to indicate that m = n is excluded. Also, $\gamma_1^2 = 4\gamma_{1h}^2/q^2$ and $\gamma_2^2 = 4\gamma_2^2/q^2$ are used.

The following abbreviations have been used,

(6.12) $$C_{mn} = \int_0^{2\pi} ce_m'(\eta)ce_n'(\eta)\ d\eta$$

(6.13) $$S_{mn} = \int_0^{2\pi} se_m'(\eta)se_n'(\eta)\ d\eta$$

(6.14) $$T_{mn} = \int_0^{2\pi} ce_m(\eta)se_n'(\eta)\ d\eta$$

(6.15) $$C_{mn}^* = \int_0^{2\pi} ce_m^{*\prime}(\eta)ce_n^{*\prime}(\eta)\ d\eta$$

(6.16) $$S_{mn}^* = \int_0^{2\pi} se_m^{*\prime}(\eta)se_n^{*\prime}(\eta)\ d\eta$$

(6.17) $$T_{mn}^* = \int_0^{2\pi} ce_m^*(\eta)se_n^{*\prime}(\eta)\ d\eta\ .$$

The power distribution characteristics for the oHE_{11} mode is given in Figure 11. The fractional power carried by the core and cladding is plotted against the normalized major axis for the various values of anisotropy and for ϵ_1/ϵ_0 = 2.5 and ξ_0 = 1.0. Most of the power is carried in the cladding near the cut-off and in the core far from cut-off. However, the difference in the power distribution for oHE_{11} for the varying anisotropy is smaller than that of eHE_{11} for a fixed value of normalized major axis.

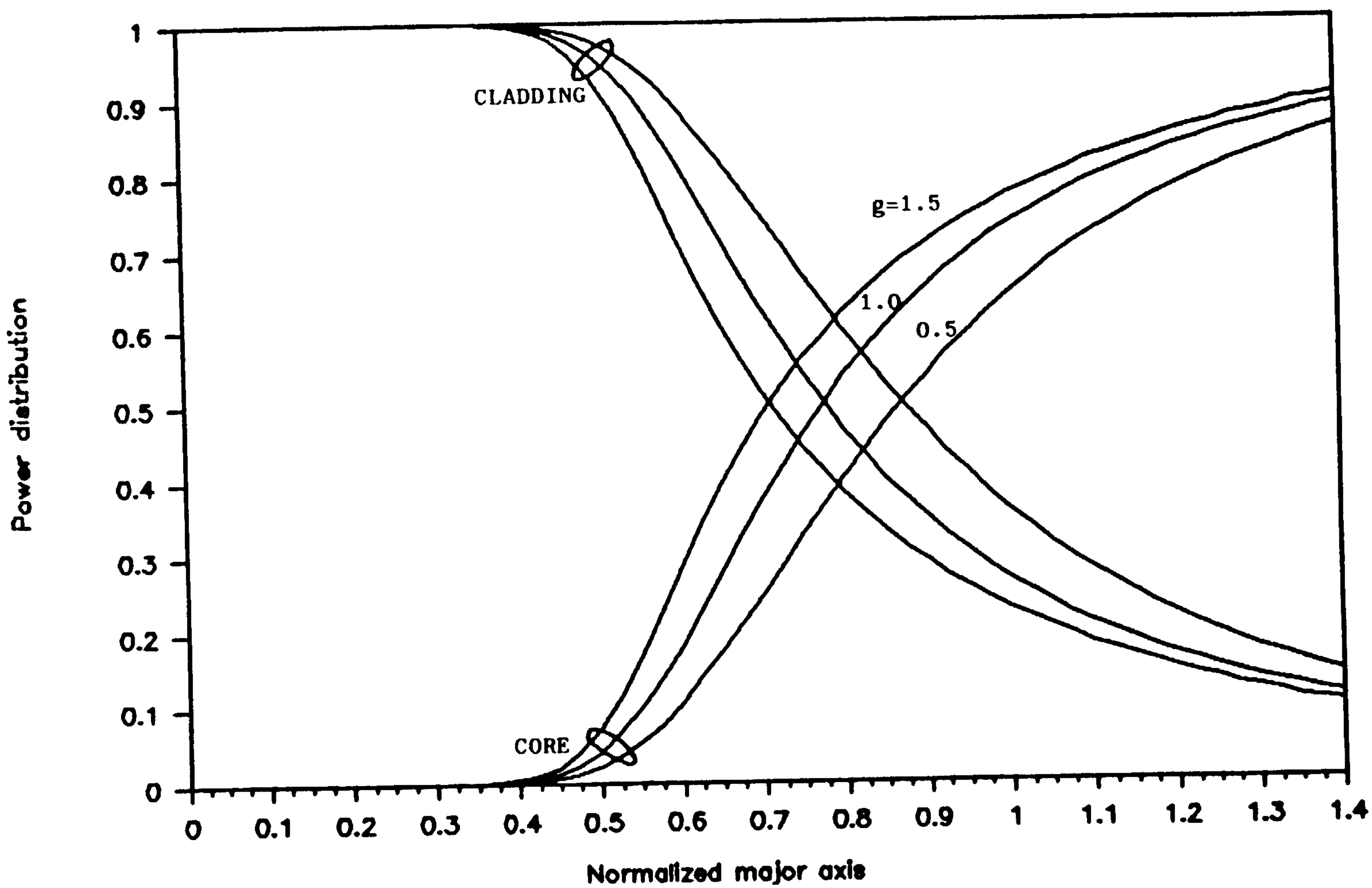


Figure 11. Power distribution characteristics for elliptical fiber as a function of normalized major axis for odd modes.

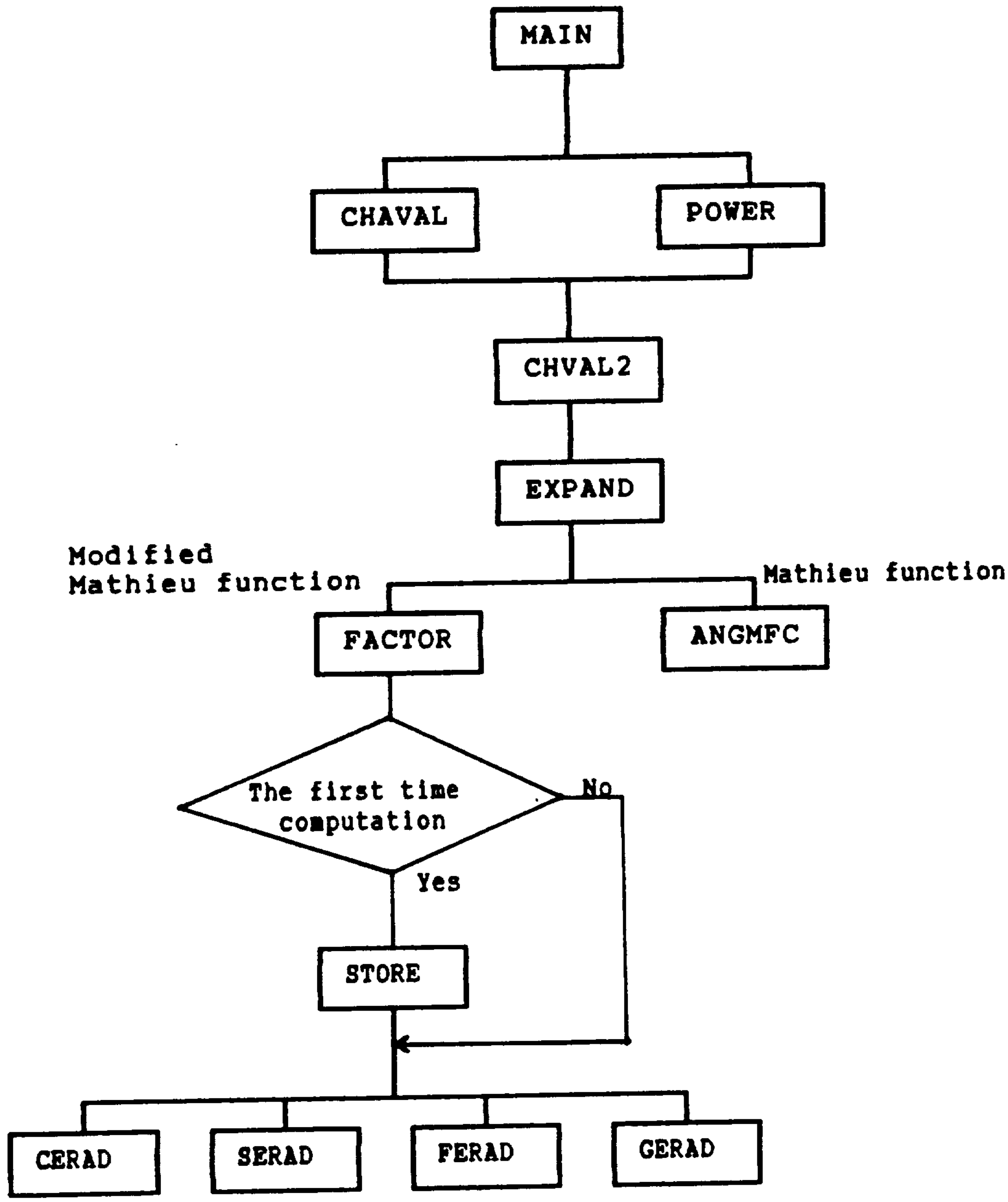
7. DESCRIPTION OF COMPUTER PROGRAMS

In this chapter, the computer programs for the anisotropic elliptical fiber will be considered. These programs are written in the language of FORTRAN IV and in order to present the complete computer programs the subroutines to calculate the Mathieu and modified Mathieu functions developed by Rengarajan and Lewis[49] will be included. The theory and notations used in the computer programs are the same as those employed by McLachlan[48].

The normalized propagation constant and power distribution characteristics as a function of the normalized cross section area or major axis for the given value of and anisotropy have been determinded by utilizing these programs. These computer programs consist of a main program and user called subroutines: CHAVAL and POWER. These subroutines CHAVAL and POWER call nine subroutines in order to compute the Mathieu and modified Mathieu functions.

In subroutine CHAVAL, an initial guess for the given mode is chosen and used to evaluate either Eq.(4.7) or Eq.(4.12). Next, Muller's method is used iteratively to determine the normalized wavelength that will minimize the function; an error criterion has been used to terminate the iteration. In subroutine POWER, the power distribution characteristics are calculated using the normalized wavelength obtained in subroutine CHAVAL. The algorithm was run on an CYBER 990 using only a single processor.

The sequence of the called subroutines is illustrated in the following flow chart.


MAIN
CHAVAL
POWER
CHVAL2
EXPAND
Modified
Mathieu function
Mathieu function
FACTOR
ANGMFC
The first time
computation
No
Yes
STORE
CERAD
SERAD
FERAD
GERAD

8. CONCLUSION

The exact characteristic equation for anisotropic elliptical optical fibers is obtained for the odd and even hybrid modes in terms of infinite determinants employing Mathieu and modified Mathieu functions. The exact characteristic equation is applicable to elliptical fibers with any ellipticity. A simplified characteristic equation can then be obtained by applying the weakly guiding approximation such that the difference in the refractive indices of the core and the cladding is small. Under this approximation, it can be shown that significant simplification can be achieved.

The simplified characteristic equation is used to compute the normalized wavelength for an anisotropic elliptical fiber. When the anisotropy parameter is equal to unity, the characteristic equation becomes that of isotropic fiber. The results are compared to the previous research and they are in close agreement. For a fixed value of the normalized cross-section area or major axis, the normalized wavelength λ/λ_0 is small for larger the value of anisotropy. This condition indicates that more energy is carried inside of the fiber. However, the geometry and anisotropy of fiber have a smaller effect when the normalized cross-section area or major axis is very small or very large.

An exact solution for the wave equation can not be determined when the thermoelastic stress causes a transverse anisotropy over the core of fibers. One possibility is that the propagation characteristics in the biaxial anisotropic fibers could be obtained by applying the numerical or approximation techniques given in Chapter 1 and this could be a subject for further study.

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APPENDIX

The following is a listing of the computer programs, MAIN, CHAVAL, POWER, CHVAL2, EXPAND, ANGMFC, FACTOR, STORE, CERAD, SERAD, FERAD AND GERAD written in FORTRAN IV language.

```
C
C     THIS PROGRAM CALULATES THE NOMALIZED WAVELENGTH AND
C     POWER DISTRIBUTION FOR ELLIPTICAL FIBER AS FUNCTION OF
C     NORMALIZED CROSS-SECTION AREA OR MAJOR AXIS.
C
      PROGRAM MAIN
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION RES(56), X(56)
      OPEN (6,FILE='OUTPUT')
C
C     NEVOD : = 1 FOR ODD MODE,
C             = 2 FOR EVEN MODE.
C     KASE  : = 1 FOR NORMALIZED CROSS-SECTIONAL AREA,
C             = 2 FOR NORMALIZED MAJOR AXIS.
C     ETA : INDEPENDENT VARIABLE IN MATHIEU FUNCTION
C     PHI : INDEPENDENT VARIABLE IN MODIFIED MATHIEU FUNCTION
C     MODE : WAVE MODE NUMBER
C     P : EP1/EPO
C     G : ANISOTROPY EPZ/EPX
C
C
      NEVOD=1
      KASE=2
      MODE=1
      P=2.5D0
      G=1.5D0
      PHI=1.0D0
      GO TO (11,12), NEVOD
11    WRITE(6,101) MODE
      GO TO 13
12    WRITE(6,105) MODE
13    WRITE(6,102) P
      WRITE(6,103) G
      WRITE(6,104) PHI
C
C     CALCULATE THE NORMALIZED WAVELENGTH
C
      CALL CHARVAL (NEVOD,P,G,MODE,PHI,KASE,RES)
C
      DO 14 I=1,56
14    X(I)=RES(I)
C
C     CALCULATE POWER DISTRIBUTION FOR THE GIVEN MODE
C
      CALL POWER (NEVOD,P,G,MODE,PHI,KASE,X)
C
101   FORMAT('1', 'THIS IS RESULT FOR ODD MODE, M =', I2)
102   FORMAT(1X, 'RATIO OF CORE AND CLADDING PERMITTIVITY =',
     *       D12.5)
103   FORMAT(1X, 'ANISOTROPY =', D12.5)
104   FORMAT(1X, 'VARIABLE IN MATHIEU FUNCTION =', D12.5)
```

```
105   FORMAT('1', 'THIS IS THE RESULT FOR EVEN MODE, M=', I2)
      STOP
      END
      SUBROUTINE CHARVAL (NEVOD,P,G,MODE,PHI,KASE,RES)
C
C     PURPOSE : CALULATE THE NORMALIZED WAVELENGTH FOR
C               ELLIPTICAL GUIDE.
C     INPUT   : NEVOD -(INTEGER) SPECIFIES = 1 FOR ODD MODE
C                                          = 2 FOR EVEN MODE
C               P -(DOUBLE PRECISION) IS THE RATIO BETWEEN CORE
C                  AND CLADDING PERMEABILITY.
C               G -(DOUBLE PRECISION) IS THE ANISOTROPY, EZ/EX.
C               MODE -(INTEGER) IS THE MODE OF CHARACTERISTIC
C                      EQUATION.
C               PHI -(DOUBLE PRECISION) IS INDEPENDENT VARIABLE
C                     IN MODIFIED MATHIEU FUNCTION.
C               KASE -(INTEGER) = 1 FOR NORMALIZED CROSS-SECTION
C                                   AREA,
C                               = 2 FOR NORMALIZED MAJOR AXIS.
C     OUTPUT  : RES -(DOUBLE PRECISION) CONTAINS THE NORMALIZED
C                     WAVELENGTH.
C
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION CHV1(23), CHV2(23), AB(25), QV(4), ETA(21),
     *          SE1(21),SE1D(21),SEO(21),CE1(21),CE1D(21),
     *          CEO(21),CEO1(21),SEO1(21),CEDSE(21),SEDCE(21),
     *          SE1SQ(21),CE1SQ(21),CEOM(21),SEOM(21),CE1M(21),
     *          SE1M(21),RES(56)
      DATA PI/3.14159265358979300/
C
C     ETA : INDEPENDENT VARIABLE IN MATHIEU FUNCTION
C     XI : INDEPENDENT VARIABLE IN MODIFIED MATHIEU FUNCTION
C     IORDER : WAVE MODE NUMBER
C     P : EP1/EPC
C     G : ANISOTROPY EPZ/EPX
C
      M=20
      LAST=40
      IF(KASE.EQ.2) LAST=56
      XI=PHI
      DT=DTANH(XI)
      DCOS2=DCOSH(XI)*DCOSH(XI)
      X2=PI*DT*DCOS2
      WRITE(6,210) X2
210   FORMAT(1X, D12.5)
C
C     SUBDIVIDE ETA FOR INTEGRATION, ALFA, BETA, GAMMA AND ANU.
C
      H=2.0D0*PI/20.0D0
      ETA(1)=0.0D0
      DO 11 I=2,21
```

```
11    ETA(I)=ETA(1)+(I-1)*H
C
C     CSA - NORMALIZED CROSS SECTION AREA OR MAJOR AXIS
C
      CSA=0.0D0
      XINC=2.5D-2
      VARX=0.9999999990D0
      DO 70 K=1,LAST
      CSA=CSA+XINC
      GO TO (12,13), KASE
12    WRITE(6,201) CSA
201   FORMAT(1X,'NORMALIZED CROSS SECTION AREA =  ',D12.5)
C
      X1=(PI*PI*CSA)/(4.0D0*DT*DCOS2)
      GO TO 14
13    WRITE(6,202) CSA
202   FORMAT(1X,'NORMALIZED MAJOR AXIS =  ',D12.5)
C
      X1=(PI*PI*CSA**2)/(4.0D0*DCOS2)
C
C     FIRST GUESS OF VAR = LANDA/LANDAO
C
14    IF(K.EQ.1) VAR=VARX
      NC=1
      NS=0
10    VAR2=1.0D0/(VAR*VAR)
C
C     EVALUATE  Q : INDEPENDANT VARIABLE
C     GAMMAE = QV(1), GAMMAH = QV(2), GAMMAZ = QV(3)
C
      QV(1)=X1*(P*G-G*VAR2)
      QV(2)=X1*(P-VAR2)
      QV(3)=X1*(1.0D0-VAR2)
      QV(4)=X1*(1.0D0-VAR2)
C
C     CALCULATE MATHIEU AND MODIFIED MATHIEU FUNCTIONS
C
      DO 50 KQ=1,4
      GO TO (15,16), NEVOD
15    IEVOD=1
      IF(MOD(KQ,2).EQ.1) IEVOD=2
      IORDER=MODE
      IF(MOD(KQ,2).EQ.1) IORDER=MODE
      Q=QV(KQ)
      CALL CHVAL2(M,Q,CHV1,CHV2,J)
      CV=CHV1(IORDER)
      IF(MOD(KQ,2).EQ.1) CV=CHV2(IORDER+1)
      GO TO 17
16    IEVOD=1
      IF(MOD(KQ,2).EQ.0) IEVOD=2
      IORDER=MODE
```

```
      IF(MOD(KQ,2).EQ.0) IORDER=MODE
      Q=QV(KQ)
      CALL CHVAL2(M,Q,CHV1,CHV2,J)
      CV=CHV1(IORDER)
      IF(MOD(KQ,2).EQ.0) CV=CHV2(IORDER+1)
C
C     OBTAIN EXPANDING COEFFICIENT, ABXX
C
17    CALL EXPAND(Q,IEVOD,IORDER,CV,3,AB,N)
C
C     CALCULATE MATHIEU FUNCTIONS AND DERIVATIVES,
C     ORDER = MODE
C
      KQEO=KQ
      IF(MOD(NEVOD,2).EQ.1) KQEO=KQ+4
      GO TO(21,22,23,24,22,21,24,23), KQEO
21    DO 41 I=1,21
      SE1(I)=ANGMFC(Q,IEVOD,IORDER,ETA(I),0,AB,N)
41    SE1D(I)=ANGMFC(Q,IEVOD,IORDER,ETA(I),1,AB,N)
      GO TO 45
23    DO 42 I=1,21
42    SEO(I)=ANGMFC(Q,IEVOD,IORDER,ETA(I),0,AB,N)
      GO TO 45
22    DO 43 I=1,21
      CE1(I)=ANGMFC(Q,IEVOD,IORDER,ETA(I),0,AB,N)
43    CE1D(I)=ANGMFC(Q,IEVOD,IORDER,ETA(I),1,AB,N)
      GO TO 45
24    DO 44 I=1,21
44    CEO(I)=ANGMFC(Q,IEVOD,IORDER,ETA(I),0,AB,N)
C
C     NORMALIZATION FACTOR FOR MODIFIED MATHIEU FUNCTION
C
45    CALL FACTOR(IEVOD,IORDER,Q,AB,N,PS)
C
C     COMPUTE AND STORE THE VALUES OF BESSEL FUNCTIONS
C
      CALL STORE(Q,XI,N)
C
C     CALCULATE MODIFIED MATHIEU FUNCTIONS
C
      GO TO (51,52,53,54,52,51,54,53),KQEO
51    SE=SERAD(Q,IORDER,0,PS,AB,N)
      SED=SERAD(Q,IORDER,1,PS,AB,N)
      GO TO 50
53    GE=GERAD(Q,IORDER,0,PS,AB,N)
      GED=GERAD(Q,IORDER,1,PS,AB,N)
      GO TO 50
52    CE=CERAD(Q,IORDER,0,PS,AB,N)
      CED=CERAD(Q,IORDER,1,PS,AB,N)
      GO TO 50
54    FE=FERAD(Q,IORDER,0,PS,AB,N)
```

```
      FED=FERAD(Q,IORDER,1,PS,AB,N)
50    CONTINUE
C
C     CALCULATE M TH TERM( = MODE) OF
C     ALFA, BETA, GAMMA AND NU.
C
      WRITE(6,107) SE,SED,GE,GED,CE,CED,FE,FED
      DO 56 I=1,21
      CEOM(I)=CEO(I)
      SEOM(I)=SEO(I)
      CE1M(I)=CE1(I)
      SE1M(I)=SE1(I)
      CEO1(I)=CEO(I)*CE1(I)
      SEO1(I)=SEO(I)*SE1(I)
      SE1SQ(I)=SE1(I)*SE1(I)
      CE1SQ(I)=CE1(I)*CE1(I)
      CEDSE(I)=CE1D(I)*SE1(I)
56    SEDCE(I)=SE1D(I)*CE1(I)
      S1=SIMPSN(CEO1,20,H)
      S2=SIMPSN(SEO1,20,H)
      S3=SIMPSN(CEDSE,20,H)
      S4=SIMPSN(SEDCE,20,H)
      S5=SIMPSN(SE1SQ,20,H)
      S6=SIMPSN(CE1SQ,20,H)
      S5M=S5
      S6M=S6
      GO TO (57,58), NEVOD
57    ALFAM=S2/S5
      BETAM=S1/S6
      GAMMAM=S4/S6
      ANUM=S3/S5
      GO TO 59
58    ALFAM=S1/S6
      BETAM=S2/S5
      GAMMAM=S3/S5
      ANUM=S4/S6
59    XMSQD=ALFAM*BETAM
      RHC=QV(2)/QV(3)
      WRITE(6,103) ALFAM,BETAM,GAMMAM,ANUM,RHC
103   FORMAT(1X,5D12.5)
C
C     CALCULATE MATHIEU FUNCTION INTEGRALS
C     ORDER = N, N+2 - -
C
      ALFA=0.0D0
      BETA=0.0D0
      GAMMA=0.0D0
      ANU=0.0D0
      XMSQN1=0.0D0
      XMSQN2=0.0D0
      DO 90 IM=1,4
```

```
      IORDER=2*IM-1
      IF(MOD(MODE,2).EQ.0) IORDER=2*IM-2
      IF(IORDER.NE.MODE) GO TO 61
      ALFA=ALFAM
      BETA=BETAM
      GAMMA=GAMMAM
      ANU=ANUM
      XMSQN1=XMSQN1+BETA*ANU
      XMSQN2=XMSQN2+ALFA*GAMMA
      XMSQ=(XMSQN1*XMSQN2)/XMSQD
      GO TO 92
61    DO 80 KQ=1,4
      GO TO (62,63), NEVOD
62    IEVOD=1
      IF(MOD(KQ,2).EQ.1) IEVOD=2
      Q=QV(KQ)
      CALL CHVAL2(M,Q,CHV1,CHV2,J)
      CV=CHV1(IORDER)
      IF(MOD(KQ,2).EQ.1) CV=CHV2(IORDER+1)
      GO TO 64
63    IEVOD=1
      IF(MOD(KQ,2).EQ.0) IEVOD=2
      Q=QV(KQ)
      CALL CHVAL2(M,Q,CHV1,CHV2,J)
      CV=CHV1(IORDER)
      IF(MOD(KQ,2).EQ.0) CV=CHV2(IORDER+1)
C
C     OBTAIN EXPANDING COEFFICIENT, ABXX
C
64    CALL EXPAND(Q,IEVOD,IORDER,CV,3,AB,N)
C
C     CALCULATE MATHIEU FUNCTIONS AND DERIVATIVES
C
      KQEO=KQ
      IF(MOD(NEVOD,2).EQ.1) KQEO=KQ+4
      GO TO(71,72,73,74,72,71,74,73), KQEO
71    DO 81 I=1,21
      SE1(I)=ANGMFC(Q,IEVOD,IORDER,ETA(I),0,AB,N)
81    SE1D(I)=ANGMFC(Q,IEVOD,IORDER,ETA(I),1,AB,N)
      GO TO 80
73    DO 82 I=1,21
82    SEO(I)=ANGMFC(Q,IEVOD,IORDER,ETA(I),0,AB,N)
      GO TO 80
72    DO 83 I=1,21
      CE1(I)=ANGMFC(Q,IEVOD,IORDER,ETA(I),0,AB,N)
83    CE1D(I)=ANGMFC(Q,IEVOD,IORDER,ETA(I),1,AB,N)
      GO TO 80
74    DO 84 I=1,21
84    CEO(I)=ANGMFC(Q,IEVOD,IORDER,ETA(I),0,AB,N)
80    CONTINUE
C
```

```
C     CALCULATE SUM (BETA * NU) AND SUM (ALFA * GAMMA)
C
      DO 86 I=1,21
      CE01(I)=CEOM(I)*CE1(I)
      SE01(I)=SEOM(I)*SE1(I)
      SE1SQ(I)=SE1(I)*SE1(I)
      CE1SQ(I)=CE1(I)*CE1(I)
      CEDSE(I)=CE1D(I)*SE1M(I)
86    SEDCE(I)=SE1D(I)*CE1M(I)
      S1=SIMPSN(CE01,20,H)
      S2=SIMPSN(SE01,20,H)
      S3=SIMPSN(CEDSE,20,H)
      S4=SIMPSN(SEDCE,20,H)
      S5=SIMPSN(SE1SQ,20,H)
      S6=SIMPSN(CE1SQ,20,H)
C
      GO TO (87,88), NEVOD
87    ALFA=S2/S5
      BETA=S1/S6
      GAMMA=S4/S6M
      ANU=S3/S5M
      GO TO 89
88    ALFA=S1/S6
      BETA=S2/S5
      GAMMA=S3/S5M
      ANU=S4/S6M
89    XMSQN1=XMSQN1+BETA*ANU
      XMSQN2=XMSQN2+ALFA*GAMMA
      XMSQ=(XMSQN1*XMSQN2)/XMSQD
92    WRITE(6,103) ALFA,BETA,GAMMA,ANU,XMSQ
90    CONTINUE
C
C     EVALUATE CHARACTERISTIC EQUATION
C
      Y1=-(XMSQ*(1.0D0-RHC)**2)
      Y2=VAR*VAR
      GO TO(93,94), NEVOD
93    Y3=(SED/SE)-(RHC*GED/GE)
      Y4=(P*CED/CE)-(RHC*FED/FE)
      WRITE(6,101) Y1, Y2, Y3, Y4
      GO TO 95
94    Y3=(CED/CE)-(RHC*FED/FE)
      Y4=(P*SED/SE)-(RHC*GED/GE)
      WRITE(6,101) Y1, Y2, Y3, Y4
95    YX=Y2*Y3*Y4
      YZ=YX/Y1
      Y=YZ-1.0D0
      WRITE(6,203) YX, Y1, YZ, Y, VAR
203   FORMAT(1X,5D12.5)
C
C     DESIDE ON TOLERANCES
```

```
C
      IF(NC.NE.1.AND.DABS(Y).LE.2.0D-3) GO TO 39
      IF(NC.EQ.1) GO TO 32
      IF(NC.EQ.2) GO TO 34
      IF(NS.EQ.0) GO TO 34
      IF(Y*YS1) 36,36,31
31    YS1=Y
      VARS1=VAR
      VAR=(VARS1+VARS2)/2.0D0
      NS=NS+1
      IF(NS.LE.20) GO TO 10
      GO TO 47
C
C     1 ST CALCULATION OF Y, DECREMENT VAR BY 0.01
C
32    YS1=Y
      VARS1=VAR
33    VAR=VAR-1.0D-2
      NC=NC+1
      YMIN=Y
      VARMIN=VAR
      IF(NC.LE.20) GO TO 10
      GO TO 48
34    IF(Y*YS1) 36,36,35
35    IF(DABS(Y)-DABS(YS1)) 32,33,33
36    YS2=Y
      VARS2=VAR
      VAR=(VARS1+VARS2)/2.0D0
      NS=NS+1
      YMIN=Y
      VARMIN=VAR
      IF(NS.LE.20) GO TO 10
47    WRITE(6,106)
      WRITE(6,108) YMIN, VARMIN
      GO TO 39
48    WRITE(6,102)
      VAR=VARX
      WRITE(6,108) YMIN, VARMIN
39    RES(K)=VAR
70    CONTINUE
      RETURN
101   FORMAT(1X,4D12.5)
102   FORMAT(1X,'ERROR : RESULT HAS SAME SIGN FOR 10 TRIES')
106   FORMAT(1X,'ERROR : 10 TRY FAILED TO OBTAIN RESOLUTION')
107   FORMAT(1X,8D12.5)
108   FORMAT(1X, 'OBTAINED RESOLUTION =', D12.5,
     C            'MIN CALCULATED NORMALIZED WAVELENGTH =', D12.5)
      END
      SUBROUTINE POWER (NEVOD,P,G,MODE,BOUND,KASE,A)
C
C     PURPOSE : CALULATE POWER DISTRIBUTION ON ELLIPTICAL
```

```
C               GUIDE.
C       INPUT   : NEVOD -(INTEGER) SPECIFIES = 1 FOR ODD MODE
C                                              = 2 FOR EVEN MODE
C               P -(DOUBLE PRECISION) IS THE RATIO BETWEEN CORE
C                  AND CLADDING PERMEABILITY.
C               G -(DOUBLE PRECISION) IS THE ANISOTROPY, EZ/EX.
C               MODE -(INTEGER) IS THE MODE OF CHARACTERISTIC
C                     EQUATION.
C               PHI -(DOUBLE PRECISION) IS INDEPENDENT VARIABLE
C                      IN MODIFIED MATHIEU FUNCTION.
C               A -(DOUBLE PRECISION) IS THE NORMALIZED
C                           WAVELENGTH.
C               KASE -(INTEGER) = 1 FOR NORMALIZED CROSS-SECTION
C                                   AREA,
C                               = 2 FOR NORMALIZED MAJOR AXIS.
C       OUTPUT  : RES -(DOUBLE PRECISION) CONTAINS THE RATIO OF
C                    POWER DISTRIBUTION.
C
        IMPLICIT DOUBLE PRECISION (A-H,O-Z)
        DIMENSION CHV1(23), CHV2(23), AB(25), QV(4), A(56),
      C          ETA(21),PHI(41),SE1(21),SE1D(21),SE0(21),
      C          SE0D(21),CE1(21),CE1D(21),CE0(21),CE0D(21),
      C          SE(21), SED(21), CE(21), CED(21),
      C          FE(41), FED(41), GE(41), GED(41),
      C          S1(21), S2(21), S3(21), S4(21), S5(21), S6(21),
      C          S7(21), S8(21), S9(21), S10(21), S11(21),
      C          S12(21), S13(21), S14(21),
      C          S21(21), S22(21), S23(21)
        DATA PI/3.14159265358979300/
C
C       ETA : INDEPENDENT VARIABLE IN MATHIEU FUNCTION
C       XI : INDEPENDENT VARIABLE IN MODIFIED MATHIEU FUNCTION
C       IORDER : WAVE MODE NUMBER
C       P : EP1/EP0
C       G : ANISOTROPY EPZ/EPX
C
        M=20
        EP=(1.0D-9)/(36.0D0*PI)
        XNU=4.0D0*PI*1.0D-7
        CONS=DSQRT(EP/XNU)
        LAST=40
        IF(KASE.EQ.2) LAST=56
        PHI0=0.0D0
        PHI1=BOUND
        PHI2=5.0D0*BOUND
        DT=DTANH(PHI1)
        DCOS2=DCOSH(PHI1)*DCOSH(PHI1)
        X2=PI*DT*DCOS2
        WRITE(6,210) X2
C
C       SUBDIVIDE ETA AND PHI FOR INTEGRATION.
```

```
C
      H1=2.0D0*PI/20.0D0
      ETA(1)=0.0D0
      DO 11 I=2,21
11    ETA(I)=ETA(I-1)+H1
      H2=BOUND/20.0D0
      PHI(1)=0.0D0
      DO 12 I=2,21
12    PHI(I)=PHI(I-1)+H2
      H3=(PHI2-PHI1)/20.0D0
      DO 13 I=22,41
13    PHI(I)=PHI(I-1)+H3
C
C     CSA - NORMALIZED CROSS SECTION AREA OR MAJOR AXIS
C
      CSA=0.0D0
      XINC=2.5D-2
      DO 70 K=1,LAST
      CSA=CSA+XINC
      IF(A(K).EQ.1.0D0) GO TO 81
      GO TO (14,15), KASE
14    WRITE(6,201) CSA
201   FORMAT(1X,'NORMALIZED CROSS SECTION AREA =  ',D12.5)
C
      X1=(PI*PI*CSA)/(4.0D0*DT*DCOS2)
      GO TO 16
15    WRITE(6,202) CSA
202   FORMAT(1X,'NORMALIZED MAJOR AXIS =  ',D12.5)
C
      X1=(PI*PI*CSA**2)/(4.0D0*DCOS2)
C
C     CALCULATE CONSTANTS.
C
16    VAR2=1.0D0/(A(K)*A(K))
C
C     EVALUATE  Q : INDEPENDANT VARIABLE
C     GAMMAE = QV(1), GAMMAH = QV(2), GAMMA2 = QV(3)
C
      QV(1)=X1*(P*G-G*VAR2)
      QV(2)=X1*(P-VAR2)
      QV(3)=X1*(1.0D0-VAR2)
      QV(4)=X1*(1.0D0-VAR2)
C
      C4=CONS/A(K)
      C1=P*C4
      C2=1.0D0/(A(K)*CONS)
      C3=P+VAR2
      C5=1.0D0+VAR2
      C6=(QV(2)*QV(2))/(QV(3)*QV(3))
C
C     CALCULATE MATHIEU AND MODIFIED MATHIEU FUNCTIONS
```

```
C
      DO 50 KQ=1,4
      GO TO (17,18), NEVOD
17    IEVOD=1
      IF(MOD(KQ,2).EQ.1) IEVOD=2
      IORDER=MODE
      IF(MOD(KQ,2).EQ.1) IORDER=MODE
      Q=QV(KQ)
      WRITE(6,301) X1,Q
      CALL CHVAL2(M,Q,CHV1,CHV2,J)
      CV=CHV1(IORDER)
      IF(MOD(KQ,2).EQ.1) CV=CHV2(IORDER+1)
      GO TO 19
18    IEVOD=1
      IF(MOD(KQ,2).EQ.0) IEVOD=2
      IORDER=MODE
      IF(MOD(KQ,2).EQ.0) IORDER=MODE
      Q=QV(KQ)
      WRITE(6,301) X1,Q
      CALL CHVAL2(M,Q,CHV1,CHV2,J)
      CV=CHV1(IORDER)
      IF(MOD(KQ,2).EQ.0) CV=CHV2(IORDER+1)
C
C     OBTAIN EXPANDING COEFFICIENT, ABXX
C
19    CALL EXPAND(Q,IEVOD,IORDER,CV,3,AB,N)
C
C     CALCULATE MATHIEU FUNCTIONS AND DERIVATIVES,
C     ORDER = MODE
C
      KQEO=KQ
      IF(MOD(NEVOD,2).EQ.1) KQEO=KQ+4
      GO TO(21,22,23,24,22,21,24,23), KQEO
21    DO 41 I=1,21
      SE1(I)=ANGMFC(Q,IEVOD,IORDER,ETA(I),0,AB,N)
41    SE1D(I)=ANGMFC(Q,IEVOD,IORDER,ETA(I),1,AB,N)
      GO TO 45
23    DO 42 I=1,21
      SEOD(I)=ANGMFC(Q,IEVOD,IORDER,ETA(I),1,AB,N)
42    SEO(I)=ANGMFC(Q,IEVOD,IORDER,ETA(I),0,AB,N)
      GO TO 45
22    DO 43 I=1,21
      CE1(I)=ANGMFC(Q,IEVOD,IORDER,ETA(I),0,AB,N)
43    CE1D(I)=ANGMFC(Q,IEVOD,IORDER,ETA(I),1,AB,N)
      GO TO 45
24    DO 44 I=1,21
      CEOD(I)=ANGMFC(Q,IEVOD,IORDER,ETA(I),1,AB,N)
44    CEO(I)=ANGMFC(Q,IEVOD,IORDER,ETA(I),0,AB,N)
C
C     NORMALIZATION FACTOR FOR MODIFIED MATHIEU FUNCTION
C
```

```
45    CALL FACTOR(IEVOD,IORDER,Q,AB,N,PS)
C
C     CALCULATE MODIFIED MATHIEU FUNCTIONS
C
      GO TO (31,32,33,34,32,31,34,33),KQEO
31    DO 51 I=1,21
      CALL STORE(Q,PHI(I),N)
      SE(I)=SERAD(Q,IORDER,0,PS,AB,N)
51    SED(I)=SERAD(Q,IORDER,1,PS,AB,N)
      GO TO 50
33    DO 52 I=21,41
      CALL STORE(Q,PHI(I),N)
      GE(I)=GERAD(Q,IORDER,0,PS,AB,N)
52    GED(I)=GERAD(Q,IORDER,1,PS,AB,N)
      GO TO 50
32    DO 53 I=1,21
      CALL STORE(Q,PHI(I),N)
      CE(I)=CERAD(Q,IORDER,0,PS,AB,N)
53    CED(I)=CERAD(Q,IORDER,1,PS,AB,N)
      GO TO 50
34    DO 54 I=21,41
      CALL STORE(Q,PHI(I),N)
      FE(I)=FERAD(Q,IORDER,0,PS,AB,N)
54    FED(I)=FERAD(Q,IORDER,1,PS,AB,N)
50    CONTINUE
C
C     CALCULATE INTEGRAND
C
      DO 56 I=1,21
      S21(I)=SE0(I)*SE1(I)
      S22(I)=SE1(I)*SE1(I)
      S23(I)=CE1D(I)*SE1(I)
      S1(I)=SE1D(I)*SE1D(I)
      S2(I)=CE1D(I)*CE1D(I)
      S3(I)=CE1D(I)*SE1(I)
      S4(I)=SE0D(I)*SE0D(I)
      S5(I)=CE0D(I)*CE0D(I)
56    S6(I)=CE0D(I)*SE0(I)
C
C
      DO 57 I=1,21
      S7(I)=SED(I)*SED(I)
      S8(I)=SE(I)*SE(I)
      S9(I)=CED(I)*CED(I)
57    S10(I)=CE(I)*CE(I)
C
C
      DO 58 I=21,41
      II=I-10
      S11(II)=GED(I)*GED(I)
      S12(II)=GE(I)*GE(I)
```

```
      S13(II)=FED(I)*FED(I)
58    S14(II)=FE(I)*FE(I)
C
C     PERFORM THE INTEGRATION
C
      ST1=SIMPSN(S21,20,H1)
      ST2=SIMPSN(S22,20,H1)
      ST3=SIMPSN(S23,20,H1)
      T11=PI*SIMPSN(S7,10,H2)
      T12=SIMPSN(S1,20,H1)*SIMPSN(S8,20,H2)
      T21=PI*SIMPSN(S9,20,H2)
      T22=SIMPSN(S2,20,H1)*SIMPSN(S10,20,H2)
      T31=SIMPSN(S3,20,H1)
      T32=(CE(21)*SE(21)-CE(1)*SE(1))
      T41=PI*SIMPSN(S11,20,H3)
      T42=SIMPSN(S4,20,H1)*SIMPSN(S12,20,H3)
      T51=PI*SIMPSN(S13,20,H3)
      T52=SIMPSN(S5,20,H1)*SIMPSN(S14,20,H3)
      T61=SIMPSN(S6,20,H1)
      T62=(FE(41)*GE(41)-FE(21)*GE(21))
C
C     CALCULATE THE ARBITRARY CONSTANTS.
C
      A11=(1.0D0-QV(2)/QV(3))*FE(21)*(ST3/ST2)
      A12=CONS*A(K)*ST1/ST2
      A13=((P*FE(21)*CED(21))/CE(21)-QV(2)*FED(21)/QV(3))
      A21=A11/(A12*A13)
      A1=A21*GE(21)/SE(21)
      A1=A1*A1
      B1=FE(21)*FE(21)
      BA1=A21*FE(21)*GE(21)/SE(21)
      A2=A21*A21
      BA2=A21
      T1=C1*A1*(T11+T12)
      T2=C2*B1*(T21+T22)
      T3=C3*BA1*T31*T32
      WRITE(6,302) T1,T2,T3
      PCOR=T1+T2-T3
      T4=C4*A2*(T41+T42)
      T5=C2*1.0D0*(T51+T52)
      T6=C5*BA2*T61*T62
      WRITE(6,302) T4,T5,T6
      PCLAD=C6*(T4+T5-T6)
      RCOR=PCOR/(PCOR+PCLAD)
      RCLAD=1.0D0-RCOR
      WRITE(6,211) RCOR, RCLAD, PCOR, PCLAD
      GO TO 70
81    WRITE(6,212)
70    CONTINUE
210   FORMAT(1X,  D12.5)
211   FORMAT(11X, 4D12.5)
```

```
212   FORMAT(1X,'NORMALIZED WAVELENGTH = 1, NO POWER CALCULATION')
301   FORMAT(1X,2D12.5)
302   FORMAT(1X,3D12.5)
      RETURN
      END
      DOUBLE PRECISION FUNCTION SIMPSN(Q,N,H)
      DOUBLE PRECISION Q(21),H
C
C     INTEGRATION BY SIMPSON'S RULE
C
      SIMPSN=Q(1)+4.0D0*Q(2)+Q(N+1)
      DO 1 I=4,N,2
1     SIMPSN=SIMPSN+2.0D0*Q(I-1)+4.0D0*Q(I)
      SIMPSN=SIMPSN*H/3.0D0
      RETURN
      END
      SUBROUTINE CHVAL2(N,QQ,CHV1,CHV2,J)
C     PURPOSE:   TO COMPUTE THE CHARACTERISTIC VALUES OF ODD
C                AND EVEN MATHIEU FUNCTIONS OF POSITIVE OR
C                NEGATIVE 'Q'
C     INPUT:     N-(INTEGER) SPECIFIES THAT CH. VALUES BE
C                OBTAINED FOR ORDERS 0 THRU N-1 FOR EVEN
C                FUNCTIONS AND FOR ORDERS 1 THRU N-1 FOR
C                ODD FUNCTIONS
C                QQ-(DOUBLE PRECISION) THE PARAMETER 'Q' IN
C                MATHIEU'S DIFFERENTIAL EQUATION
C     OUPUT:     CHV1-(DOUBLE PRECISION) AN ARRAY OF LENGTH N
C                CONTAINING CH. VALUES OF ODD MATHIEU FUNCTIONS
C                OF ORDERS 1 THRU N-1.
C                CHV1(N) IS A DUMMY VARIABLE.
C                CHV2-(DOUBLE PRECISION) AN ARRAY OF LENGTH N
C                CONTAINING CH. VALUES OF EVEN MATHIEU FUNCTIONS
C                OF ORDERS 0 THRU N-1.
C                J-(INTEGER) MAXIMUM ORDER UPTO WHICH CH. VALUES
C                HAVE BEEN SUCCESSFULLY COMPUTED
      DOUBLE PRECISION CV1(6,25),CV2(6,25),CHV1(N),CHV2(N),QQ
      DOUBLE PRECISION QABS,DABS
101   FORMAT('0',5X,'NOT ALL CH. VALUES AVAILABLE----WARNING')
      IF(QQ.LT.0.D0) GO TO 20
      IF(N.GT.1) CALL MFCVAL(N-1,N-1,QQ,CV1,J1)
      CALL MFCVAL(N,N-1,QQ,CV2,J2)
      IF(J1.LT.(N-1).OR.J2.LT.N) WRITE(6,101)
      J=MIN0(J1,J2-1)
      DO 10 I=1,J
        IF(N.GT.1) CHV1(I) =CV1(1,I)
        CHV2(I)=CV2(1,I)
10    CONTINUE
      CHV2(J+1)=CV2(1,J+1)
      RETURN
20    QABS=DABS(QQ)
      CALL MFCVAL(N,N-1,QABS,CV2,J2)
```

```
      IF(N.NE.1) GO TO 25
      CHV2(1)=CV2(1,1)
      RETURN
25    CALL MFCVAL(N-1,N-1,QABS,CV1,J1)
      IF(J1.NE.(N-1).OR.J2.NE.N) WRITE(6,101)
      J=MIN0(J1,J2-1)
      DO 30 I=1,J,2
        CHV2(I)=CV2(1,I)
        CHV1(I)=CV2(1,I+1)
        CHV2(I+1)=CV1(1,I)
        IF((I+1).LE.J)CHV1(I+1)=CV1(1,I+1)
30    CONTINUE
      IF(MOD((N-1),2).EQ.0)CHV2(N)=CV2(1,N)
      RETURN
      END
      SUBROUTINE MFCVAL(N,R,QQ,CV,J)
C     **********
      INTEGER J,K,KK,L,M,N,R,TYPE
      DOUBLE PRECISION A,CV,DL,DR,DTM,Q,QQ,T,TM,TOL,TOLA
      DOUBLE PRECISION FILL(3)
      DIMENSION CV(6,N)
      EQUIVALENCE (DL,DR,T)
      COMMON/MF1/Q,TOL,TYPE,DUMMY(4)
      COMMON/MF2/FILL
      TOL=1.0D-13
      IF(N-R) 10,10,20
10    L=1
      GO TO 30
20    L=2
30    Q=QQ
      DO 500 K=1,N
      J=K
      IF(Q) 960,490,40
40    KK=MIN0(K,4)
      TYPE=2*MOD(L,2)+MOD(K-L+1,2)
C     FIRST APPROXIMATION
      GO TO(100,200,300,400),KK
100     IF(Q-1.0D0)110,140,140
110       GO TO(120,130),L
120       A=1.0D0-Q-.125D0*Q*Q
          GO TO 420
130       A=Q*Q
          A=A*(-.5D0+.0546875D0*A)
          GO TO 420
140     IF(Q-2.0D0) 150,180,180
150       GO TO(160,170),L
160       A=1.033D0-1.0746D0*Q-.0688D0*Q*Q
          GO TO 420
170       A=.23D0-.495D0*Q-.191D0*Q*Q
          GO TO 420
180       A=-.25D0-2.0D0*Q+2.0D0*DSQRT(Q)
```

```
            GO TO 420
200       DL=L
          IF(Q*DL-6.0D0) 210,350,350
210         GO TO(220,230),L
220         A=4.01521D0-Q*(.046D0+.0667857D0*Q)
            GO TO 420
230         A=1.0D0+1.05007D0*Q-.180143D0*Q*Q
            GO TO 420
300       IF(Q-8.0D0) 310,350,350
310         GO TO(320,330),L
320         A=8.93867D0+.178156D0*Q-.0252132D0*Q*Q
            GO TO 420
330         A=3.70017D0+.953485D0*Q-.0475065D0*Q*Q
            GO TO 420
350         DR=K-1
            A=CV(1,K-1)-DR+4.0D0*DSQRT(Q)
            GO TO 420
400       A=CV(1,K-1)-CV(1,K-2)
          A=3.0D0*A+CV(1,K-3)
420     IF(Q.GE.1.0D0) GO TO 440
        IF(K.NE.1) GO TO 430
425     TOLA=DMAX1(DMIN1(TOL,DABS(A)),1.0D-14)
        GO TO 450
430     TOLA=TOL*DABS(A)
        GO TO 450
440     TOLA=TOL*DMAX1(Q,DABS(A))
445     TOLA=DMAX1(DMIN1(TOLA,DABS(A),.4D0*DSQRT(Q)),1.0D-14)
C       CRUDE UPPER AND LOWER BOUNDS
450     CALL BOUNDS(K,A,TOLA,CV,N,M)
        IF(M.NE.0) IF(M-1) 470,910,900
C       ITERATE
        CALL MFITR8(TOLA,CV(1,K),CV(2,K),M)
        IF(M.GT.0) GO TO 920
C       FINAL BOUNDS AND FUNCTIONS, D
470     T=CV(1,K)-TOLA
        CALL TMOFA(T,TM,DTM,M)
        IF(M.GT.0) GO TO 940
        CV(3,K)=T
        CV(4,K)=-TM/DTM
480     T=CV(1,K)+TOLA
        CALL TMOFA(T,TM,DTM,M)
        IF(M.GT.0) GO TO 950
        CV(5,K)=T
        CV(6,K)=-TM/DTM
        GO TO 500
C       Q EQUALS ZERO
490     CV(1,K)=(K-L+1)**2
        CV(2,K)=0.0D0
        CV(3,K)=CV(1,K)
        CV(4,K)=0.0D0
        CV(5,K)=CV(1,K)
```

```
      CV(6,K)=0.0D0
500   CONTINUE
550   RETURN
C     PRINT ERROR MESSAGES
900   WRITE(6,901) K
901   FORMAT('0','CRUDE BOUNDS CANNOT',' BE LOCATED, NO OUTPUT',
     C       ' FOR K=',I2)
      GO TO 930
910   WRITE(6,911) K
911   FORMAT('0','ERROR IN SUBPROGRAM  TMOFA, VIA SUBPROGRAM
     C          BOUNDS, NO OUTPUT, FOR K=',I2)
      GO TO 930
920   WRITE(6,921) K
921   FORMAT('0','ERROR IN SUBPROGRAM, TMOFA, VIA SUBPROGRAM,
     C          MFITR8, NO OUTPUT, FOR K=',I2)
930   J=J-1
      GO TO 550
940   WRITE(6,941) K
941   FORMAT('0','ERROR IN SUBPROGRAM, TMOFA, NO LOWER BOUND,
     C          FOR K=',I2)
      CV(3,K)=0.0D0
      CV(4,K)=0.D0
      GO TO 480
950   WRITE(6,951) K
951   FORMAT('0','ERROR IN SUBPROGRAM, TMOFA, NO UPPER BOUND,
     C          FOR K=',I2)
      CV(5,K)=0.D0
      CV(6,K)=0.D0
      GO TO 500
960   WRITE(6,961)
961   FORMAT(20H0Q GIVEN NEGATIVELY,,20H USED ABSOLUTE VALUE)
      Q=-Q
      GO TO 40
      END
      SUBROUTINE BOUNDS(K,APPROX,TOLA,CV,N,MM)
      INTEGER K,KA,M,MM,N
      DOUBLE PRECISION A,APPROX,A0,A1,CV,DTM,D0,D1,Q,TM,TOLA
      DIMENSION CV(6,N)
      COMMON/MF1/Q,DUMMY(7)
      COMMON/MF2/A0,A,A1
      KA=0
      IF(K.EQ.1) GO TO 20
      IF(APPROX-CV(1,K-1)) 10,10,20
10    A0=CV(1,K-1)+1.0D0
      GO TO 30
20    A0=APPROX
30    CALL TMOFA(A0,TM,DTM,M)
      IF(M.GT.0) GO TO 250
      D0=-TM/DTM
      IF(D0) 100,300,50
C     A0 IS LOWER BOUND,
```

```
C       SEARCH FOR UPPER BOUND
50      A1=A0+D0+.1D0
        CALL TMOFA(A1,TM,DTM,M)
        IF(M.GT.0) GO TO 250
        D1=-TM/DTM
        IF(D1) 200,350,60
60      A0=A1
        D0=D1
        KA=KA+1
        IF(KA-4) 50,400,400
C       A1 IS UPPER BOUND, SEARCH FOR LOWER BOUND
100     A1=A0
        D1=D0
        A0=DMAX1(A1+D1-.1D0,-2.0D0*Q)
        IF(K.EQ.1) GO TO 110
        IF(A0-CV(1,K-1)) 150,150,110
110     CALL TMOFA(A0,TM,DTM,M)
        IF(M.GT.0) GO TO 250
        D0=-TM/DTM
        IF(D0) 120,300,200
120     KA=KA+1
        IF(KA-4) 100,400,400
150     KA=KA+1
        IF(KA-4) 160,400,400
160     A0=A1+DMAX1(TOLA,DABS(D1))
        GO TO 30
200     A=.5D0*(A0+D0+A1+D1)
        IF(A.LE.A0.OR.A.GE.A1) A=.5D0*(A0+A1)
250     MM=M
        RETURN
300     CV(1,K)=A0
310     CV(2,K)=0.D0
        M=-1
        GO TO 250
350     CV(1,K)=A1
        GO TO 310
400     M=2
        GO TO 250
        END
        SUBROUTINE MFITRB(TOLA,CV,DCV,MM)
        INTEGER M,MM,N
        DOUBLE PRECISION A,A0,A1,A2,CV,D,DCV,DTM,TM,TOLA
        LOGICAL LAST
        COMMON/MF2/A0,A,A1
        N=0
        LAST=.FALSE.
50      N=N+1
        CALL TMOFA(A,TM,DTM,M)
        IF(M.GT.0) GO TO 400
        D=-TM/DTM
C       IS TOLERANCE MET
```

```
      IF(N.EQ.40.OR.A-AC.LE.TOLA.OR.A1-A.LE.TOLA.OR.
     C   DABS(D).LT.TOLA)LAST=.TRUE.
      IF(D) 110,100,120
100   CV=A
      DCV=0.DO
      GO TO 320
C     REPLACE UPPER BOUND BY A
110   A1=A
      GO TO 200
C     REPLACE LOWER BOUND BY A
120   A0=A
200   A2=A+D
      IF(LAST) GO TO 300
      IF(A2.GT.A0.AND.A2.LT.A1) GO TO 250
      A=.5D0*(A0+A1)
      GO TO 50
250   A=A2
      GO TO 50
300   IF(A2.LE.A0.OR.A2.GE.A1) GO TO 350
      CALL TMOFA(A2,TM,DTM,M)
      IF(M.GT.0) GO TO 400
      D=-TM/DTM
      CV=A2
310   DCV=D
320   MM=M
      RETURN
350   CV=A
      GO TO 310
400   CV=0.DO
      DCV=0.DO
      GO TO 320
      END
      SUBROUTINE TMOFA(ALFA,TM,DTM,ND)
      INTEGER K,KK,KT,L,MF,MO,M1,M2S,ND,TYPE
      DOUBLE PRECISION A,AA,ALFA,B,DG,DTM,DTYPE,
     C                 F,FL,G,H(200),HP,Q,QINV,
     C                 Q1,Q2,T,TM,TOL,TT,V
      COMMON G(200,2),DG(200,2),AA,A(3),B(3),DTYPE,QINV,
     C       Q1,Q2,T,TT,K,L,KK,KT
      COMMON/MF1/Q,TOL,TYPE,M1,MO,M2S,MF
      EQUIVALENCE (H(1),G(1,1)),(Q1,HP),(Q2,F)
      DATA FL/1.OD+30/
C     STATEMENT FUNCTION
      V(K)=(AA-DBLE(FLOAT(K))**2)/Q
      ND=0
      KT=0
      AA=ALFA
      DTYPE=TYPE
      QINV=1.ODC/Q
      DO 10 L=1,2
        DO 5 K=1,200
```

```
          G(K,L)=0.D0
          DG(K,L)=0.D0
5         CONTINUE
10      CONTINUE
        IF(MOD(TYPE,2)) 20,30,20
20      MO=3
        GO TO 40
30      MO=TYPE+2
40      K=.5D0+DSQRT(DMAX1(3.0D0*Q+AA,0.D0))
        M2S=MIN0(2*K+MO+4,398+MOD(MO,2))
C       EVALUATION OF THE TAIL OF A CONTINUED FRACTION
        A(1)=1.0D0
        A(2)=V(M2S+2)
        B(1)=V(M2S)
        B(2)=A(2)*B(1)-1.0D0
        Q1=A(2)/B(2)
        DO 50 K=1,200
        MF=M2S+2+2*K
        T=V(MF)
        A(3)=T*A(2)-A(1)
        B(3)=T*B(2)-B(1)
        Q2=A(3)/B(3)
        IF(DABS(Q1-Q2).LT.TOL) GO TO 70
        Q1=Q2
        A(1)=A(2)
        A(2)=A(3)
        B(1)=B(2)
        B(2)=B(3)
50      CONTINUE
        KT=1
70      T=1.0D0/T
        TT=-T*T*QINV
        L=MF-M2S
        DO 80 K=2,L,2
        T=1.0D0/(V(MF-K)-T)
        TT=T*T*(TT-QINV)
80      CONTINUE
        KK=M2S/2+1
        IF(KT.EQ.1) Q2=T
        G(KK,2)=.5D0*(Q2+T)
        DG(KK,2)=TT
C       STAGE 1
        G(2,1)=1.0D0
        DO 140 K=MO,M2S,2
        KK=K/2+1
        IF(K.LT.5) IF(K-3) 100,110,120
        G(KK,1)=V(K-2)-1.0D0/G(KK-1,1)
        DG(KK,1)=QINV+DG(KK-1,1)/G(KK-1,1)**2
        GO TO 130
100     G(2,1)=V(0)
        DG(2,1)=QINV
```

```
      GO TO 130
110   G(2,1)=V(1)+DTYPE-2.0D0
      DG(2,1)=QINV
      GO TO 130
120   G(3,1)=V(2)+(DTYPE-2.D0)/G(2,1)
      DG(3,1)=QINV+(2.D0-DTYPE)*DG(2,1)/G(2,1)**2
      IF(TYPE.EQ.2) G(2,1)=0.D0
130   IF(DABS(G(KK,1)).LT.1.D0) GO TO 200
140   CONTINUE
C     BACKTRACK
      TM=G(KK,2)-G(KK,1)
      DTM=DG(KK,2)-DG(KK,1)
      M1=M2S
      KT=M2S-M0
      DO 180 L=2,KT,2
      K=M2S-L
      KK=K/2+1
      G(KK,2)=1.D0/(V(K)-G(KK+1,2))
      DG(KK,2)=-G(KK,2)**2*(QINV-DG(KK+1,2))
      IF(K-2) 150,150,160
150   G(2,2)=2.0D0*G(2,2)
      DG(2,2)=2.D0*DG(2,2)
160   T=G(KK,2)-G(KK,1)
      IF(DABS(T)-DABS(TM)) 170,180,180
170   TM=T
      DTM=DG(KK,2)-DG(KK,1)
      M1=K
180   CONTINUE
      GO TO 320
C     STAGE 2
200   M1=K
      K=M2S
      KK=K/2+1
210   IF(K.EQ.M1) IF(K-2) 300,300,310
      K=K-2
      KK=KK-1
      T=V(K)-G(KK+1,2)
      IF(DABS(T)-1.D0) 250,220,220
220   G(KK,2)=1.0D0/T
      DG(KK,2)=(DG(KK+1,2)-QINV)/T**2
      GO TO 210
C     STAGE 3
250   IF(K.EQ.M1) IF(T) 220,290,220
      HP=DG(KK+1,2)-QINV
260   G(KK,2)=FL
      H(KK)=T
      K=K-2
      KK=KK-1
      F=V(K)*T-1.D0
      IF(K.EQ.M1) IF(F) 280,290,280
      IF(DABS(F)-DABS(T)) 270,280,280
```

```
270   HP=HP/T**2-QINV
      T=F/T
      GO TO 260
280   G(KK,2)=T/F
      DG(KK,2)=(HP-QINV*T*T)/F**2
      GO TO 210
290   ND=1
      GO TO 320
C     CHAINING M EQUALS 2
300   G(2,2)=2.D0*G(2,2)
      DG(2,2)=2.D0*DG(2,2)
310   TM=G(KK,2)-G(KK,1)
      DTM=DG(KK,2)-DG(KK,1)
320   RETURN
      END
      SUBROUTINE EXPAND(QD,FNC,R,CV,NORM,CD,N)
C     PURPOSE:  TO GET EXPANDING COEFFICIENTS FROM ROUTINE
C               COEF.  TERMINATE THE TERMS FOR REQD. ACCUTACY
C               AND DO THE NORMALIZATION
      DIMENSION CD(25)
      DOUBLE PRECISION A,CV,QD,Q,TOL,T,AB,ERR,DABS,
     C                 CD,SUM,T1,DSQRT,SUM1
      INTEGER R,FNC,TYPE,CASE,NORM
      COMMON DUM1(1600),A,T,DUM2(6),AB(200)
      COMMON /MF1/Q,TOL,TYPE,M1,MD,M25,MF
101   FORMAT('0','THE # OF EXPANDING COEFFICIENT REQD. IS MORE
     C  THAN 25',5X,'WARNING')
102   FORMAT('0','ERROR IN SUBPROGRAM','TMOFA VIA COEF. VERIFY
     C  ARGUMENTS  NO OUTPUT')
      TOL=1.0D-13
C     TO TEST THE LAST VALUE OF ARRAY CD ERR IS USED
      ERR=1.0D-20
      Q=QD
      TYPE=2*MOD(FNC,2)+MOD(R,2)
C     FOR NEGATIVE Q AND ODD ORDERS, EXP. COEFFS. FOR EVEN AND
C     ODD FUNCTIONS ARE INTERCHANGED.
      IF(Q.LT.0.0D0.AND.MOD(R,2).EQ.1)TYPE=2*MOD((FNC-1),2)+MOD(R,2)
      M=0
      A=CV
      Q=DABS(Q)
      CALL COEF(M)
      IF(M.EQ.0) GO TO 5
      WRITE(6,102)
      RETURN
5     TYPE=2*MOD(FNC,2)+MOD(R,2)
      CASE=TYPE+1
C     THE COEFFICIENTS PASSED THRU COMMON ARRAY AB IN DOUBLE
C     PRECISION  IS GIVEN TO AN ARRAY CD OF LENGTH 25 FOR
C     FURTHER PROCESSION
      DO 10 I=1,25
      CD(I)=AB(I)
```

```
      IF(DABS(CD(I)).LT.1.D-30) CD(I)=0.D0
10    CONTINUE
      IF(CD(25).GT.ERR) WRITE(6,101)
      N1=R/2+1
      DO 20 I=N1,25
      IF(CD(I).EQ.0.D0) GO TO 25
      N=I
20    CONTINUE
C     NORMALISING THE CODES.  PRESENTLY IN NEUTRAL NORM
25    SUM=0.0D0
      IF(NORM.EQ.1) GO TO 140
C     GETTING STRATTON NORMALISATION FACTOR
      IF(QD.LT.0.D0) GO TO 91
      GO TO (40,40,60,80),CASE
40    DO 50 J=1,N
      SUM=SUM+CD(J)
50    CONTINUE
      GO TO 100
60    DO 70 J=1,N
      SUM=SUM+CD(J)*DBLE(FLOAT(2*(J-1)))
70    CONTINUE
      GO TO 100
80    DO 90 J=1,N
      SUM=SUM+CD(J)*DBLE(FLOAT(2*J-1))
90    CONTINUE
      GO TO 100
C     GOT NEGATIVE Q STRATTON NORMALISATION FACTOR IS DIFFERENT
91    T1=-1.0D0
      IF(MOD(R/2,2).EQ.1)T1=-T1
      GO TO(92,92,94,96),CASE
92    DO 93 J=1,N
      T1=-T1
      SUM=SUM+T1*CD(J)
93    CONTINUE
      GO TO 100
94    DO 95 J=1,N
      T1=-T1
      SUM=SUM+CD(J)*T1*DBLE(FLOAT(2*(J-1)))
95    CONTINUE
      GO TO 100
96    DO 97 J=1,N
      T1=-T1
      SUM=SUM+CD(J)*T1*DBLE(FLOAT(2*J-1))
97    CONTINUE
100   IF(NORM.EQ.2) GO TO 120
C     GETTING INCE'S NORMALISATION FACTOR
      SUM1=0.D0
      DO 110 J=1,N
      SUM1=SUM1+CD(J)*CD(J)
110   CONTINUE
      IF(FNC.EQ.2.AND.MOD(R,2).EQ.0) SUM1=SUM1+CD(1)*CD(1)
```

```
      SUM1=DSQRT(SUM1)
      IF(NORM.EQ.3) SUM=DSIGN(SUM1,SUM)
C     DIVIDE ALL COEFS. BY NORMALISING FACTOR FOR 2 & 3 ONLY
120    DO 130 I=1,N
      CD(I)=CD(I)/SUM
130   CONTINUE
C     FOR MATHIEU FUNCTIONS OF SE2N+2 TYPE(CASE=3) COEFS. SHOULD
C     BE B2, B4  ETC. BUT THE ROUTINE COEF RETURNS A B0=0 ALSO.
C     THIS IS TO BE DROPPED.
140   IF(CASE.NE.3) RETURN
      DO 150 I=2,N
      CD(I-1)=CD(I)
150   CONTINUE
      CD(N)=0.D0
      RETURN
      END
      SUBROUTINE COEF(M)
      INTEGER  K,KA,KB,KK,M,MF,ML,MM,MO,M1,M2S,TYPE
      DOUBLE PRECISION  A,AB,FL,G,H(200),Q,T,TOL,V,V2
      COMMON  G(200,2),DUM1(800),A,T,K,KA,KB,KK,MM,ML,AB(200)
      COMMON /MF1/Q,TOL,TYPE,M1,MO,M2S,MF
      EQUIVALENCE (H(1),G(1,1))
      DATA FL,V2/1.D+30,1.D-15/
C     STATEMENT FUNCTION
      V(K)=(A-DBLE(FLOAT(K))**2)/Q
      CALL TMOFA(A,T,T,M)
      IF(M.NE.0) GO TO 300
      DO 60 K=1,200
         AB(K)=0.D0
60    CONTINUE
      KA=M1-MO+2
      DO 90 K=2,KA,2
         KK=(M1-K)/2+1
         IF(K-2) 70,70,80
70       AB(KK)=1.D0
         GO TO 90
80       AB(KK)=AB(KK+1)/G(KK+1,1)
90    CONTINUE
      KA=0
      DO 130 K=M1,M2S,2
         KK=K/2+1
         ML=K
         IF(G(KK,2).EQ.FL) GO TO 100
         AB(KK)=AB(KK-1)*G(KK,2)
         GO TO 110
100      T=AB(KK-2)
         IF(K.EQ.4.AND.M1.EQ.2) T=T+T
         AB(KK)=T/(V(K-2)*H(KK)-1.D0)
110      IF(DABS(AB(KK)).GE.1.D-17) KA=0
         IF(KA.EQ.5) GO TO 260
         KA=KA+1
```

```
130   CONTINUE
      T=DLOG(DABS(AB(KK))/V2)/DLOG(1.D0/DABS(G(KK,2)))
      KA=2*IDINT(T)
      ML=KA+2+M2S
      IF(ML.GT.399) GO TO 400
      KB=KA+2+MF
      T=1.D0/V(KB)
      KK=MF-M2S
      DO 150 K=2,KK,2
         T=1.D0/(V(KB-K)-T)
150   CONTINUE
      KK=ML/2+1
      G(KK,2)=T
      DO 200 K=2,KA,2
         KK=(ML-K)/2+1
         G(KK,2)=1.D0/(V(ML-K)-G(KK+1,2))
200   CONTINUE
      KA=M2S+2
      DO 250 K=KA,ML,2
         KK=K/2+1
         AB(KK)=AB(KK-1)*G(KK,2)
250   CONTINUE
C     NEUTRAL NORMALIZATION
260   T=AB(1)
      MM=MOD(TYPE,2)
      KA=MM+2
      DO 280 K=KA,ML,2
         KK=K/2+1
         IF(DABS(T)-DABS(AB(KK))) 270,280,280
270   T=AB(KK)
      MM=K
280   CONTINUE
      DO 290 K=1,KK
         AB(K)=AB(K)/T
290   CONTINUE
300   RETURN
400   M=-1
      GO TO 300
      END
      DOUBLE PRECISION FUNCTION ANGMFC(QD,FNC,R,XD,DERIV,AR,N)
C     PURPOSE:  TO COMPUTE A PERIODIC MATHIEU FUNCTION,
C               ODD OR EVEN TYPE OR ITS DERIVATIVE
      DOUBLE PRECISION PC,PS,DPC,DPS
      EXTERNAL PC,PS,DPC,DPS
      DIMENSION AR(25),AB(25)
      INTEGER FNC,R,DERIV,TYPE,CASE,P
      DOUBLE PRECISION AR,AB,XD,X,T1,SUM,DCOS,DSIN,QD
      COMMON/NTERM/NLIMIT
      COMMON/ANG/AB,X,P
      NLIMIT=N
      QS=QD
```

```
      X=XD
      TYPE=2*MOD(FNC,2)+MOD(R,2)
      CASE=TYPE+1
      DO 1 I=1,N
      AB(I)=AR(I)
1     CONTINUE
C     FOR NEGATIVE Q IN ALL SUMMATIONS ALTERNATE TERMS HAVE
C     A MINUS SIGN
      IF(QS)20,90,35
20    T1=-1.0D0
      IF(CASE.EQ.3)T1=1.0D0
      IF(MOD(R/2,2).EQ.1)T1=-T1
      DO 30 I=1,N
      T1=-T1
      AB(I)=T1*AB(I)
30    CONTINUE
35    P=-1
      IF(CASE.EQ.1) P=-2
      IF(CASE.EQ.3) P=0
      IF(DERIV.EQ.1) GO TO 60
      GO TO(40,40,50,50),CASE
40    CALL SIGMA(PC,SUM)
      ANGMFC=SUM
      RETURN
50    CALL SIGMA(PS,SUM)
      ANGMFC=SUM
      RETURN
60    GO TO(70,70,80,80),CASE
70    CALL SIGMA(DPC,SUM)
      ANGMFC=SUM
      RETURN
80    CALL SIGMA(DPS,SUM)
      ANGMFC=SUM
      RETURN
90    IF(DERIV.EQ.1)GO TO 120
      GO TO(100,100,110,110),CASE
100   ANGMFC=DCOS(DBLE(FLOAT(R))*X)
      RETURN
110   ANGMFC=DSIN(DBLE(FLOAT(R))*X)
      RETURN
120   GO TO(130,130,140,140),CASE
130   ANGMFC=-R*DSIN(DBLE(FLOAT(R))*X)
      RETURN
140   ANGMFC=R*DCOS(DBLE(FLOAT(R))*X)
      RETURN
      END
      DOUBLE PRECISION FUNCTION PC(K)
      INTEGER P,K
      DOUBLE PRECISION AB(25),X,DCOS
      COMMON/ANG/AB,X,P
C     EVALUATES ONE TERM OF THE EVEN PERIODIC SOLUTION
```

```
      PC=AB(K)*DCOS(DBLE(FLOAT(2*K+P))*X)
      RETURN
      END
      DOUBLE PRECISION FUNCTION PS(K)
      INTEGER P,K
      DOUBLE PRECISION AB(25),X,DSIN
      COMMON/ANG/AB,X,P
C     EVALUATES ONE TERM OF THR ODD PERIODIC SOLUTION
      PS=AB(K)*DSIN(DBLE(FLOAT(2*K+P))*X)
      RETURN
      END
      DOUBLE PRECISION FUNCTION DPC(K)
      INTEGER P,K
      DOUBLE PRECISION AB(25),X,T,DSIN
      COMMON/ANG/AB,X,P
C     EVALUATES ONE TERM OF THE DERIVATIVE OF THE EVEN PERIODIC
C     MATHIEU FUNCTION.
      T=2*K+P
      DPC=-AB(K)*T*DSIN(T*X)
      RETURN
      END
      DOUBLE PRECISION FUNCTION DPS(K)
      INTEGER P,K
      DOUBLE PRECISION AB(25),X,T,DCOS
      COMMON/ANG/AB,X,P
C     EVALUATES ONE TERM OF THE DERIVATIVE OF THE ODD PERIODIC
C     MATHIEU FUNCTION.
      T=2*K+P
      DPS=AB(K)*T*DCOS(T*X)
      RETURN
      END
      SUBROUTINE SIGMA(DUM,SUM)
C     PURPOSE:  TO SUM N TERMS (SPECIFIED BY THE COMMON
C               COMMON BLOCK N TERM) OF A FUNCTION.
      DOUBLE PRECISION DUM,SUM,ERR,T1,TERM,DABS
      COMMON/NTERM/NLIMIT
101   FORMAT('C','CONVERGENCE NOT TO SATISFACTION',10X,'WARNING')
      ERR=1.0D-13
      T1=DUM(1)
      SUM=T1
      N=NLIMIT
      IF(NLIMIT.GE.22)N=22
      IMIN=5
      DO 10 I=2,N
      TERM=DUM(I)
      SUM=SUM+TERM
      TERM=DABS(TERM)
      IF(I.LT.IMIN) GO TO 10
        IF(DABS(SUM).GE.ERR*TERM) RETURN
10    CONTINUE
20    IF(I.EQ.22)WRITE(6,101)
```

```
      RETURN
      END
      SUBROUTINE FACTOR(FNC,R,QD,AB,N,PS)
C     PURPOSE:  TO COMPUTE THE NORMALIZATION FACTOR,P2N,P2N+1,
C               S2N+1,OR S2N+2 FOR MODIFIED MATHIEU FUNCTIONS
C               OF POSITIVE 'Q' AND P2N',P2N+1',S2N+1',
C               OR S2N+2' FOR THOSE OF NEGATIVE 'Q'
      DIMENSION AB(25)
      INTEGER FNC,R,CASE
      DOUBLE PRECISION AB,PS,PSP,DABS,DSQRT,RQ,ODEV,
     C                 SUM1,SUM2,T1,T2,QD
      RQ=DSQRT(DABS(QD))
      CASE=2*MOD(FNC,2)+MOD(R,2)+1
      IF(QD.LT.0.0D0)CASE=CASE+4
      ODEV=1.0D0
      IF(MOD(R/2,2).EQ.1)ODEV=-1.0D0
      SUM1=0.0D0
      SUM2=0.0D0
      T1=-1.0D0
      GO TO (10,30,50,70,10,70,50,30),CASE
C     FOR ALL Q AND EVEN ORDER ---- P2N AND P2N' ---- FNC=2
10    DO 20 I=1,N
      SUM1=SUM1+AB(I)
      T1=-T1
      SUM2=SUM2+T1*AB(I)
20    CONTINUE
      PS=SUM1*SUM2/AB(1)
      PSP=PS*ODEV
      IF(QD.LT.0.0D0)PS=PSP
      RETURN
C     FOR POSITIVE Q AND ODD ORDERS P2N+1, P2N+1'  IF FNC=2
C       NEG Q AND ODD ORDER IF FNC=1
30    DO 40 I=1,N
      T1=-T1
      SUM1=SUM1+AB(I)
      SUM2=SUM2+T1*AB(I)*DBLE(FLOAT(2*I-1))
40    CONTINUE
      PS=SUM1*SUM2/(RQ*AB(1))
      PSP=PS*ODEV
      IF(QD.LT.0.D0) PS=PSP
      RETURN
C     FOR ALL Q AND EVEN ORDER IF FNC=1 ---- S2N+2, S2N+2'
50    T1=1.0D0
      DO 60 I=1,N
      T2=AB(I)*DBLE(FLOAT(2*I))
      SUM1=SUM1+T2
      T1=-T1
      SUM2=SUM2+T1*T2
60    CONTINUE
      PS=SUM1*SUM2/(RQ*RQ*AB(1))
      PSP=PS*ODEV
```

```
      IF(QD.LT.0.0D0)PS=PSP
      RETURN
C     FOR POSITIVE Q AND ODD ORDER ---- IF FNC=1
C     FOR NEGATIVE Q AND ODD ORDERS S2N+1, S2N+1'    IF FNC=2
70    DO 80 I=1,N
      SUM1=SUM1+AB(I)*DBLE(FLOAT(2*I-1))
      T1=-T1
      SUM2=SUM2+T1*AB(I)
80    CONTINUE
      PS=SUM1*SUM2/(RQ*AB(1))
      PSP=PS*ODEV
      IF(QD.LT.0.0D0)PS=PSP
      RETURN
      END
      SUBROUTINE STORE(QD,XI,NMAX)
C     PURPOSE:  TO COMPUTE AND STORE THE VALUES OF BESSEL
C               FUNCTIONS AND DERIVATIVES REQUIRED IN MATHIEU
C               FUNCTION CALCULATION.
      DOUBLE PRECISION QD,QABS,RQ,DABS,DSQRT,DEXP,BS1V1,BS1V2,
     *                 BS2V2,DBS1V1,DBS1V2,DBS2V2,XI,V1,V2
      COMMON/LOCAL/DUMMY1(4),V1,V2,DUMMY2(50)
      COMMON/RADIAL/BS1V1(25),BS1V2(25),BS2V2(25),DBS1V1(25),
     *       DBS1V2(25),DBS2V2(25)
      N=NMAX+3
      IF(N.GE.25)N=25
      N1=N-1
      QABS=DABS(QD)
      RQ=DSQRT(QABS)
      V1=RQ*DEXP(-XI)
      V2=QABS/V1
      IF(QD.LT.0.0D0) GO TO 20
      CALL BESSEL(1,V1,BS1V1,N)
      CALL BESSEL(1,V2,BS1V2,N)
      CALL BESSEL(2,V2,BS2V2,N)
      DBS1V1(1)=-BS1V1(2)
      DBS1V2(1)=-BS1V2(2)
      DBS2V2(1)=-BS2V2(2)
      DO 10 I=2,N1
      DBS1V1(I)=(BS1V1(I-1)-BS1V1(I+1))*0.5D0
      DBS1V2(I)=(BS1V2(I-1)-BS1V2(I+1))*0.5D0
      DBS2V2(I)=(BS2V2(I-1)-BS2V2(I+1))*0.5D0
10    CONTINUE
      RETURN
20    CALL BSL2(1,V1,BS1V1,N)
      CALL BSL2(1,V2,BS1V2,N)
      CALL BSL2(2,V2,BS2V2,N)
      DBS1V1(1)=BS1V1(2)
      DBS1V2(1)=BS1V2(2)
      DBS2V2(1)=-BS2V2(2)
      DO 30 I=2,N1
      DBS1V1(I)=(BS1V1(I-1)+BS1V1(I+1))*0.5D0
```

```
      DBS1V2(I)=(BS1V2(I-1)+BS1V2(I+1))*0.5D0
      DBS2V2(I)=-(BS2V2(I-1)+BS2V2(I+1))*0.5D0
30    CONTINUE
      RETURN
      END
      SUBROUTINE BESSEL(SOL,U,BSJY,N)
      INTEGER N,NN,SOL
      DOUBLE PRECISION BSJY(N),U
      NN=N-1
      IF(U.EQ.0.D0.AND.SOL.EQ.2) GO TO 80
      IF(U.GE.8.D0) GO TO 30
      GO TO(10,20),SOL
10    CALL J0J1(U,BSJY)
      GO TO 40
20    CALL Y0Y1(U,BSJY)
      GO TO 40
30    CALL LUKE(U,SOL,BSJY)
40    IF(N.LT.2) GO TO 100
      GO TO(50,60),SOL
50    CALL JNS(BSJY,U,N)
      GO TO 100
C     RECURRENCE FORMULAR
60    DO 70 K=2,NN
      BSJY(K+1)=2.D0*DBLE(FLOAT(K-1))*BSJY(K)/U-BSJY(K-1)
70    CONTINUE
      GO TO 100
80    NN=NN+1
      DO 90 K=1,NN
      BSJY(K)=-1.D+37
90    CONTINUE
100   RETURN
      END
      SUBROUTINE J0J1(X,BJ)
      DOUBLE PRECISION BJ(2),T(5),X
      T(1)=X/2.D0
      BJ(1)=1.D0
      BJ(2)=T(1)
      T(2)=-T(1)**2
      T(3)=1.D0
      T(4)=1.D0
10    T(4)=T(4)*T(2)/T(3)**2
      BJ(1)=BJ(1)+T(4)
      T(5)=T(4)*T(1)/(T(3)+1.D0)
      BJ(2)=BJ(2)+T(5)
      IF(DMAX1(DABS(T(4)),DABS(T(5))).LT.1.D-15) RETURN
      T(3)=T(3)+1.D0
      GO TO 10
      END
      SUBROUTINE Y0Y1(X,BY)
      DOUBLE PRECISION T(10),X,BY(2)
      T(1)=X/2.D0
```

```
      T(2)=-T(1)**2
      BY(1)=1.D0
      BY(2)=T(1)
      T(7)=0.D0
      T(10)=-T(1)
      T(3)=0.D0
      T(4)=0.D0
      T(5)=1.D0
   10 T(3)=T(3)+1.D0
      T(4)=T(4)+1.D0/T(3)
      T(5)=T(5)*T(2)/T(3)**2
      BY(1)=BY(1)+T(5)
      T(6)=-T(5)*T(4)
      T(7)=T(7)+T(6)
      T(8)=T(5)*T(1)/(T(3)+1.D0)
      BY(2)=BY(2)+T(8)
      T(9)=-T(8)*(2.D0*T(4)+1.D0/(T(3)+1.D0))
      T(10)=T(10)+T(9)
      IF(DMAX1(DABS(T(6)),DABS(T(9))).GE.1.D-15) GO TO 10
      T(2)=.57721566490153286D0+DLOG(T(1))
      BY(1)=.63661977236758134D0*(BY(1)*T(2)+T(7))
      BY(2)=.63661977236758134D0*(BY(2)*T(2)-1.D0/X)+T(10)/
     C      3.14159265358979320D0
      RETURN
      END
      SUBROUTINE LUKE(U,KIND,BSJY)
      INTEGER K,KIND
      DOUBLE PRECISION A(19),B(19),CS,C(19),D(19),G(3),BSJY(2),
     C                 R(2),S(2),SN,T,U,X
C     WARNING - THE FOLLOWING DATA STATEMENTS ARE NOT IN ASA
C               STANDARD FORTRAN
      DATA A/.99959506476996728741 6D0,
     *       -.53807956139606913D-3,
     *       -.13179677123361570D-3,
     *        .151422497048644D-5,
     *        .15846861792063D-6,
     *       -.856069553946D-8,
     *       -.29572343355D-9,
     *        .6573556254D-10,
     *       -.2237497030D-11,
     *       -.4482114CD-12,
     *        .6954827D-13,
     *       -.151340D-14,
     *       -.92422D-15,
     *        .15558D-15,
     *       -.476D-17,
     *       -.274D-17,
     *        .61D-18,
     *       -.4D-19,
     *       -.1D-19/
      DATA B/-.77693556942053213 6D-2,
```

```
*      -.77480323096544767OD-2,
*       .25365411654307960D-4,
*       .39427359839971lD-5,
*      -.10723498299129D-6,
*      -.72138979932 8D-8,
*       .73764602893D-9,
*       .150687811D-11,
*      -.574589537D-11,
*       .45996574D-12,
*       .2270323D-13,
*      -.887890D-14,
*       .74497D-15,
*       .5847D-16,
*      -.2410D-16,
*       .265D-17,
*       .13D-18,
*      -.10D-18,
*       .2D-19/
 DATA C/1.00067753586591346234D0,
*       .90100725195908183D-3,
*       .22172434918599454D-3,
*      -.19657594631910 4D-5,
*      -.20889531143 27D-6,
*       .10281443508 94D-7,
*       .37597054789D-9,
*      -.7638891358D-10,
*       .23873467 0D-11,
*       .51825489D-12,
*      -.7693969D-13,
*       .144008D-14,
*       .103294D-14,
*      -.16821D-15,
*       .459D-17,
*       .302D-17,
*      -.65D-18,
*       .4D-19,
*       .1D-19/
 DATA D/.23376829986285803280-1,
*       .23346801223545575330-1,
*      -.35760105909013820-4,
*      -.56086314949262 7D-5,
*       .13273894084340D-6,
*       .91697584506 6D-8,
*      -.86838880371D-9,
*      -.3780730035D-11,
*       .663145586D-11,
*      -.50584390D-12,
*      -.2720782D-13,
*       .985381D-14,
*      -.79398D-15,
*      -.6757D-16,
```

```
     *        .2625D-16,
     *       -.280D-17,
     *       -.15D-18,
     *        .10D-18,
     *       -.2D-19/
      X=8.D0/U
      G(1)=1.D0
      G(2)=2.D0*X-1.D0
      R(1)=A(1)+A(2)*G(2)
      S(1)=B(1)+B(2)*G(2)
      R(2)=C(1)+C(2)*G(2)
      S(2)=D(1)+D(2)*G(2)
      DO 10 K=3,19
      G(3)=(4.D0*X-2.D0)*G(2)-G(1)
      R(1)=R(1)+A(K)*G(3)
      S(1)=S(1)+B(K)*G(3)
      R(2)=R(2)+C(K)*G(3)
      S(2)=S(2)+D(K)*G(3)
      G(1)=G(2)
      G(2)=G(3)
10    CONTINUE
      T=.79788456080286540D/DSQRT(U)
      SN=DSIN(U-.785398163397448300)
      CS=DCOS(U-.785398163397448300)
      GO TO(20,30),KIND
20    BSJY(1)=T*(R(1)*CS-S(1)*SN)
      BSJY(2)=T*(R(2)*SN+S(2)*CS)
      GO TO 40
30    BSJY(1)=T*(S(1)*CS+R(1)*SN)
      BSJY(2)=T*(S(2)*SN-R(2)*CS)
40    RETURN
      END
      SUBROUTINE JNS(JJ,U,M)
      INTEGER K,KA,KK,M
      DOUBLE PRECISION A,B,D(2),DM,G(25),JJ(M),
     C                    P(3),Q(3),U
      DM=2*M
      P(1)=0.D0
      Q(1)=1.D0
      P(2)=1.D0
      Q(2)=DM/U
      D(1)=P(2)/Q(2)
      A=2.D0
10    B=(DM+A)/U
      P(3)=B*P(2)-P(1)
      Q(3)=B*Q(2)-Q(1)
      D(2)=P(3)/Q(3)
      IF(DABS(D(1)-D(2)).LT.1.D-15) GO TO 20
      P(1)=P(2)
      P(2)=P(3)
      Q(1)=Q(2)
```

```
      Q(2)=Q(3)
      D(1)=D(2)
      A=A+2.D0
      GO TO 10
20    G(M)=D(2)
      KA=M-2
      DO 30 K=1,KA
      KK=M-K
      A=2*KK
      G(KK)=U/(A-U*G(KK+1))
      IF(G(KK).EQ.0.D0) G(KK)=1.D-35
30    CONTINUE
      DO 40 K=2,M
      JJ(K+1)=G(K)*JJ(K)
40    CONTINUE
      RETURN
      END
      SUBROUTINE BSL2(SOL,U,BSIK,N)
C     PURPOSE:  TO COMPUTE MODIFIED BESSEL FUNCTIONS, 'I' OR 'K'
C               TYPE FOR ORDERS 0 THRU N-1 IN DOUBLE PRECISION.
      INTEGER N,SOL
      DOUBLE PRECISION BSIK(N),U
      IF(SOL.EQ.2)GO TO 30
      IF(U.GE.8.0D0) GO TO 10
      CALL IOI1(U,BSIK)
      GO TO 20
10    CALL LUKE2(U,SOL,BSIK)
20    IF(N.LT.2)RETURN
      CALL INS(BSIK,U,N)
      RETURN
30    IF(U.EQ.0.0D0) GO TO 70
      IF(U.GE.5.0D0) GO TO 40
      CALL KOK1(U,BSIK)
      GO TO 50
40    CALL LUKE2(U,SOL,BSIK)
50    IF(N.LT.2)RETURN
C     RECURRENCE FORMULA
      NN=N-1
      DO 60 K=2,NN
      BSIK(K+1)=2.0D0*DBLE(FLOAT(K-1))*BSIK(K)/U+BSIK(K-1)
60    CONTINUE
      RETURN
70    DO 80 K=1,N
      BSIK(K)=1.0D+75
80    CONTINUE
      RETURN
      END
      SUBROUTINE IOI1(X,BI)
C     PURPOSE: TO EVALUATE 'I0' AND 'I1' BESSEL FUNCTIONS
C              BY SUMMING THE SERIES.
      DOUBLE PRECISION BI(2),T(5),X,DMAX1,DABS
```

```
      T(1)=X/2.0D0
      BI(1)=1.0D0
      BI(2)=T(1)
      T(2)=T(1)**2
      T(3)=1.0D0
      T(4)=1.0D0
   10 T(4)=T(4)*T(2)/T(3)**2
      BI(1)=BI(1)+T(4)
      T(5)=T(4)*T(1)/(T(3)+1.0D0)
      BI(2)=BI(2)+T(5)
      IF(DMAX1(DABS(T(4)),DABS(T(5))).LT.1.0D-15)RETURN
      T(3)=T(3)+1.0D0
      GO TO 10
      END
      SUBROUTINE KOK1(X,BK)
C     PURPOSE:  TO EVALUATE 'K0' AND 'K1' BESSEL FUNCTIONS
C                   BY SUMMING THE SERIES.
      DOUBLE PRECISION T(10),X,BK(2),DMAX1,DABS,DLOG
      T(1)=X/2.0D0
      T(2)=T(1)**2
      BK(1)=1.0D0
      BK(2)=T(1)
      T(7)=0.0D0
      T(10)=-T(1)
      T(3)=0.0D0
      T(4)=0.0D0
      T(5)=1.0D0
   10 T(3)=T(3)+1.0D0
      T(4)=T(4)+1.0D0/T(3)
      T(5)=T(5)*T(2)/T(3)**2
      BK(1)=BK(1)+T(5)
      T(6)=T(5)*T(4)
      T(7)=T(7)+T(6)
      T(8)=T(5)*T(1)/(T(3)+1.0D0)
      BK(2)=BK(2)+T(8)
      T(9)=-T(8)*(2.0D0*T(4)+1.0D0/(T(3)+1.0D0))
      T(10)=T(10)+T(9)
      IF(DMAX1(DABS(T(6)),DABS(T(9))).GE.1.0D-15) GO TO 10
      T(2)=.57721566490153290D0+DLOG(T(1))
      BK(1)=-BK(1)*T(2)+T(7)
      BK(2)=BK(2)*T(2)+1.0D0/X+T(10)/2.0D0
      RETURN
      END
      SUBROUTINE LUKE2(U,KIND,BSIK)
C     PURPOSE:  TO EVALUATE MODIFIED BESSEL FUNCTIONS, I0 AND I1
C                   OR K0 AND K1 FROM SHIFTED CHEBYSHEV SERIES.
      DOUBLE PRECISION A(34),B(21),C(34),D(21),U,BSIK(2),X,G(34),
     C                   R(2),S(2),DEXP,DSQRT
      DATA A/1.0082792054587400,.84451226249209430-2,
     *.17270063077756650-3,.72475910999589 60-5,.51358772687802D-6,
     *.56816965809120-7,.85130912228 5D-8,.1238425364D-8,
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```
*.298016723D-10,-.7895669832D-10,-.33127127630-10,
*-.4497338640-11,.17997903D-11,.965748320-12,.3860424D-13,
*-.10403934D-12,-.2395045D-13,.955447D-14,.444315D-14,
*-.85864D-15,-.70878D-15,.8676D-16,.11194D-15,-.1211D-16,
*-.1813D-16,.249D-17,.299D-17,-.62D-18,-.49D-18,.16D-18,
*.7D-19,-.4D-19,-.1D-19,.1D-19/
 DATA B/.98840917423082580D0,-.1131050446469282D-1,
*.2695326127627237D-3,-.1110668519666535D-4,
*.632575108500490D-6,-.4504733764110D-7,.37929964556 8D-8,
*-.36454717921D-9,.3904375576D-10,-.4579936220-11,
*.590810630-12,-.7883236D-13,.1136042D-13,
*-.172697D-14,.27545D-15,-.4589D-16,.796D-17,-.143D-17,
*.27D-18,-.5D-19,.1D-19/
 DATA C/.97580060232628590D0,-.24467442963276380D-1,
*-.27720536076382890D-3,-.97321467280201 3D-5,
*-.62972423863981D-6,-.6596114215424D-7,-.96138729194D-8,
*-.14011409010 3D-8,-.4756316654D-10,.81530681070D-10,
*.35408148320-10,.51025640 7D-11,-.1804409340-11,
*-.102359447D-11,-.526778 4D-13,.10709419D-12,.2611976D-13,
*-.956129D-14,-.471335D-14,.82924D-15,.74262D-15,-.8045D-16,
*-.11657D-15,.1107D-16,.1884D-16,-.233D-17,-.311D-17,
*.61D-18,.51D-18,-.16D-18,-.8D-19,.4D-19,.1D-19,-.1D-19/
 DATA D/1.03595085877235800,.3546529124333111D-1,
*-.46847502816688860-3,.16185063810053430-4,
*-.84517204812368D-6,.57132218102840-7,-.46455546066610-8,
*.43541733857D-9,-.45757297040-10,.5288132810-11,
*-.66261293D-12,.89047920-13,-.1272607D-13,.192086D-14,
*-.3045D-15,.5045D-16,-.871D-17,.156D-17,-.29D-18,
*.6D-19,-.1D-19/
  IF(KIND.EQ.2) GO TO 20
  X=8.0D0/U
  G(1)=1.0D0
  G(2)=2.0D0*X-1.0D0
  N=34
  DO 10 K=3,N
  G(K)=(4.0D0*X-2.0D0)*G(K-1)-G(K-2)
10 CONTINUE
  R(1)=0.0D0
  R(2)=0.0D0
  DO 15 K=1,N
  I=N+1-K
  R(1)=R(1)+A(I)*G(I)
  R(2)=R(2)+C(I)*G(I)
15 CONTINUE
  BSIK(1)=.3989422804014327D0*R(1)*DEXP(U)/DSQRT(U)
  BSIK(2)=.3989422804014327D0*R(2)*DEXP(U)/DSQRT(U)
  RETURN
20 X=5.0D0/U
  G(1)=1.0D0
  G(2)=2.0D0*X-1.0D0
  N=21
```

```
      DO 25 K=3,N
      G(K)=(4.0D0*X-2.0D0)*G(K-1)-G(K-2)
   25 CONTINUE
      S(1)=0.0D0
      S(2)=0.0D0
      DO 30 K=1,N
      I=N+1-K
      S(1)=S(1)+B(I)*G(I)
      S(2)=S(2)+D(I)*G(I)
   30 CONTINUE
      BSIK(1)=1.2533141373155D0*S(1)*DEXP(-U)/DSQRT(U)
      BSIK(2)=1.2533141373155D0*S(2)*DEXP(-U)/DSQRT(U)
      RETURN
      END
      SUBROUTINE INS(II,U,M)
C     PURPOSE:  TO EVALUATE 'I' BESSEL FUNCTIONS OF HIGHER
C               ORDERS BY A CONTINUED FRACTION EXPANSION
C               METHODS.
      INTEGER K,KA,KK,M
      DOUBLE PRECISION A,B,D(2),DM,G(25),II(M),P(3),Q(3),U
      IF(U.EQ.0.0D0) GO TO 50
      DM=2*M
      P(1)=0.0D0
      Q(1)=1.0D0
      P(2)=1.0D0
      Q(2)=DM/U
      D(1)=P(2)/Q(2)
      A=2.0D0
   10 B=(DM+A)/U
      P(3)=B*P(2)+P(1)
      Q(3)=B*Q(2)+Q(1)
      D(2)=P(3)/Q(3)
      IF(DABS(D(1)-D(2)).LT.1.0D-15) GO TO 20
      P(1)=P(2)
      P(2)=P(3)
      Q(1)=Q(2)
      Q(2)=Q(3)
      D(1)=D(2)
      A=A+2.0D0
      GO TO 10
   20 G(M)=D(2)
      KA=M-2
      DO 30 K=1,KA
      KK=M-K
      A=2*KK
      G(KK)=U/(A+U*G(KK+1))
      IF(DABS(G(KK)).LE.1.0D-35) G(KK)=1.0D-35
   30 CONTINUE
      DO 40 K=2,M
      IF(DABS(II(K)).LT.1.0D-35) GO TO 35
      II(K+1)=G(K)*II(K)
```

```
      GO TO 40
35    II(K+1)=0.0D0
40    CONTINUE
      RETURN
50    DO 60 I=3,M
60    II(I)=0.0D0
      RETURN
      END
      DOUBLE PRECISION FUNCTION CERAD(QD,R,DERIV,PS,AR,N)
C     PURPOSE:  TO COMPUTE A MODIFIED MATHIEU FUNCTION
C               (OR DERIVATIVE) OF FIRST KIND CORRESPONDING
C               TO EVEN MATHIEU FUNCTION ( CE FUNCTIONS)
      EXTERNAL C2NP,C2NIP,C2NN,C2NIN,DC2NP,DC2NIP,DC2NN,DC2NIN
      DOUBLE PRECISION AB(25),QD,PS,PSP,OUTPUT,AR(25)
      INTEGER R,CASE,DERIV,FNC
      COMMON/NTERM/N1
      COMMON/LOCAL/DUMMY(8),AB
      N1=N
      PSP=PS
      DO 5 I=1,N
      AB(I)=AR(I)
5     CONTINUE
      CASE=MOD(R,2)+1
      IF(QD.LT.0.0D0) CASE=CASE+2
      IF(DERIV.EQ.1) GO TO 50
      GO TO(10,20,30,40),CASE
C     THE VALUE OF CE2N(Z,Q)
10    CALL SIGMA(C2NP,OUTPUT)
      CERAD=PS*OUTPUT/AB(1)
      RETURN
C     THE VALUE OF CE2N+1(Z,Q)
20    CALL SIGMA(C2NIP,OUTPUT)
      CERAD=PS*OUTPUT/AB(1)
      RETURN
C     THE VALUE OF CE2N(Z,-Q)
30    CALL SIGMA(C2NN,OUTPUT)
      CERAD=PSP*OUTPUT/AB(1)
      RETURN
C     THE VALUE OF C2EN+1(Z,-Q)
40    CALL SIGMA(C2NIN,OUTPUT)
      CERAD=PSP*OUTPUT/AB(1)
      RETURN
C     FOLLOWING ARE DERIVATIVES OF FUNCTIONS
50    GO TO(60,70,80,90),CASE
C     THE VALUE OF CE2N'(Z,Q)
60    CALL SIGMA(DC2NP,OUTPUT)
      CERAD=PS*OUTPUT/AB(1)
      RETURN
C     THE VALUE OF CE2N+1'(Z,Q)
70    CALL SIGMA(DC2NIP,OUTPUT)
      CERAD=PS*OUTPUT/AB(1)
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      RETURN
C     THE VALUE OF CE2N'(Z,-Q)
80    CALL SIGMA(DC2NN,OUTPUT)
      CERAD=PSP*OUTPUT/AB(1)
      RETURN
C     THE VALUE OF CE2N+1'(Z,-Q)
90    CALL SIGMA(DC2N1N,OUTPUT)
      CERAD=PSP*OUTPUT/AB(1)
      RETURN
      END
      DOUBLE PRECISION FUNCTION C2NP(K)
C     PURPOSE:  TO CALCULATE K TH TERM IN SUM OF SERIES
C               FOR CE2N(Z,Q).
      DOUBLE PRECISION AB(25),BSJV1,BSJV2,BSYV2,DBSJV1,DBSJV2,
     *                DBSYV2
      COMMON/LOCAL/DUMMY(8),AB
      COMMON/RADIAL/BSJV1(25),BSJV2(25),BSYV2(25),DBSJV1(25),
     C              DBSJV2(25),DBSYV2(25)
      C2NP=AB(K)*BSJV1(K)*BSJV2(K)
      IF(MOD(K,2).EQ.0)C2NP=-C2NP
      RETURN
      END
      DOUBLE PRECISION FUNCTION C2N1P(K)
C     PURPOSE:  TO CALCULATE K TH TERM IN SUM OF SERIES
C               FOR CE2N+1(Z,Q).
      DOUBLE PRECISION AB(25),BSJV1,BSJV2,BSYV2,DBSJV1,DBSJV2,
     *                DBSYV2
      COMMON/LOCAL/DUMMY(8),AB
      COMMON/RADIAL/BSJV1(25),BSJV2(25),BSYV2(25),DBSJV1(25),
     C              DBSJV2(25),DBSYV2(25)
      C2N1P=AB(K)*(BSJV1(K)*BSJV2(K+1)+BSJV1(K+1)*BSJV2(K))
      IF(MOD(K,2).EQ.0)C2N1P=-C2N1P
      RETURN
      END
      DOUBLE PRECISION FUNCTION C2NN(K)
C     PURPOSE:  TO CALCULATE K TH TERM IN SUM OF SERIES
C               FOR CE2N(Z,-Q).
      DOUBLE PRECISION AB(25),BSIV1,BSIV2,BSKV2,DBSIV1,DBSIV2,
     *                DBSKV2
      COMMON/LOCAL/DUMMY(8),AB
      COMMON/RADIAL/BSIV1(25),BSIV2(25),BSKV2(25),DBSIV1(25),
     *              DBSIV2(25),DBSKV2(25)
      C2NN=AB(K)*BSIV1(K)*BSIV2(K)
      IF(MOD(K,2).EQ.0)C2NN=-C2NN
      RETURN
      END
      DOUBLE PRECISION FUNCTION C2N1N(K)
C     PURPOSE:  TO CALCULATE K TH TERM IN SUM OF SERIES FOR CE2N+1(Z,-Q)
      DOUBLE PRECISION AB(25),BSIV1,BSIV2,BSKV2,DBSIV1,DBSIV2,DBSKV2
      COMMON/LOCAL/DUMMY(8),AB
      COMMON/RADIAL/BSIV1(25),BSIV2(25),BSKV2(25),DBSIV1(25),DBSIV2(25),
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C                   DBSKV2(25)
      C2NIN=AB(K)*(BSIV1(K)*BSIV2(K+1)+BSIV1(K+1)*BSIV2(K))
      IF(MOD(K,2).EQ.0)C2NIN=-C2NIN
      RETURN
      END
      DOUBLE PRECISION FUNCTION DC2NP(K)
C     PURPOSE:  TO CALCULATE K TH TERM IN SUM OF SERIES
C               FOR CE2N'(Z,Q).
      DOUBLE PRECISION AB(25),BSJV1,BSJV2,BSYV2,DBSJV1,DBSJV2,
     *                 DBSYV2,V1,V2
      COMMON/LOCAL/DUMMY1(4),V1,V2,AB
      COMMON/RADIAL/BSJV1(25),BSJV2(25),BSYV2(25),DBSJV1(25),
     *              DBSJV2(25),DBSYV2(25)
      DC2NP=AB(K)*(-DBSJV1(K)*BSJV2(K)*V1+BSJV1(K)*DBSJV2(K)*V2)
      IF(MOD(K,2).EQ.0)DC2NP=-DC2NP
      RETURN
      END
      DOUBLE PRECISION FUNCTION DC2NIP(K)
C     PURPOSE:  TO CALCULATE K TH TERM IN SUM OF SERIES
C               FOR CE2N+1'(Z,Q).
      DOUBLE PRECISION AB(25),BSJV1,BSJV2,BSYV2,DBSJV1,DBSJV2,
     *                 DBSYV2,V1,V2
      COMMON/LOCAL/DUMMY1(4),V1,V2,AB
      COMMON/RADIAL/BSJV1(25),BSJV2(25),BSYV2(25),DBSJV1(25),
     *              DBSJV2(25),DBSYV2(25)
      DC2NIP=AB(K)*(-DBSJV1(K)*BSJV2(K+1)*V1+
     *              BSJV1(K)*DBSJV2(K+1)*V2-
     *       DBSJV1(K+1)*BSJV2(K)*V1+BSJV1(K+1)*DBSJV2(K)*V2)
      IF(MOD(K,2).EQ.0)DC2NIP=-DC2NIP
      RETURN
      END
      DOUBLE PRECISION FUNCTION DC2NN(K)
C     PURPOSE:  TO CALCULATE K TH TERM IN SUM OF SERIES
C               FOR CE2N'(Z,Q).
      DOUBLE PRECISION AB(25),BSIV1,BSIV2,BSKV2,DBSIV1,DBSIV2,
     *                 DBSKV2,V1,V2
      COMMON/LOCAL/DUMMY1(4),V1,V2,AB
      COMMON/RADIAL/BSIV1(25),BSIV2(25),BSKV2(25),DBSIV1(25),
     *              DBSIV2(25),DBSKV2(25)
      DC2NN=AB(K)*(-DBSIV1(K)*BSIV2(K)*V1+BSIV1(K)*DBSIV2(K)*V2)
      IF(MOD(K,2).EQ.0)DC2NN=-DC2NN
      RETURN
      END
      DOUBLE PRECISION FUNCTION DC2NIN(K)
C     PURPOSE:  TO CALCULATE K TH TERM IN SUM OF SERIES
C               FOR CE2N+1'(Z,-Q).
      DOUBLE PRECISION AB(25),BSIV1,BSIV2,BSKV2,DBSIV1,DBSIV2,
     *                 DBSKV2,V1,V2
      COMMON/LOCAL/DUMMY1(4),V1,V2,AB
      COMMON/RADIAL/BSIV1(25),BSIV2(25),BSKV2(25),DBSIV1(25),
     *              DBSIV2(25),DBSKV2(25)
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      DC2NIN=AB(K)*(-DBSIV1(K)*BSIV2(K+1)*V1+
     *               BSIV1(K)*DBSIV2(K+1)*V2-
     *      DBSIV1(K+1)*BSIV2(K)*V1+BSIV1(K+1)*DBSIV2(K)*V2)
      IF(MOD(K,2).EQ.0)DC2NIN=-DC2NIN
      RETURN
      END
      DOUBLE PRECISION FUNCTION SERAD(QD,R,DERIV,PS,AR,N)
C     PURPOSE:  TO COMPUTE A MODIFIED MATHIEU FUNCTION
C               (OR DERIVATIVE) OF FIRST KIND CORRESPONDING TO
C               ODD MATHIEU FUNCTION ( SE FUNCTIONS)
      EXTERNAL S2N2P,S2N1P,S2N2N,S2N1N,DS2N2P,DS2N1P,DS2N2N,
     *         DS2N1N
      DOUBLE PRECISION AB(25),QD,PS,PSP,OUTPUT,AR(25)
      INTEGER R,CASE,DERIV
      COMMON/NTERM/N1
      COMMON/LOCAL/DUMMY(8),AB
      N1=N
      PSP=PS
      DO 5 I=1,N
      AB(I)=AR(I)
5     CONTINUE
      CASE=MOD(R,2)+1
      IF(QD.LT.0.0D0) CASE=CASE+2
      IF(DERIV.EQ.1) GO TO 50
      GO TO(10,20,30,40),CASE
C     THE VALUE OF SE2N+2(Z,Q)
10    CALL SIGMA(S2N2P,OUTPUT)
      SERAD=-PS*OUTPUT/AB(1)
      RETURN
C     THE VALUE OF SE2N+1(Z,Q)
20    CALL SIGMA(S2N1P,OUTPUT)
      SERAD=PS*OUTPUT/AB(1)
      RETURN
C     THE VALUE OF SE2N+2(Z,-Q)
30    CALL SIGMA(S2N2N,OUTPUT)
      SERAD=PSP*OUTPUT/AB(1)
      RETURN
C     THE VALUE OF S2EN+1(Z,-Q)
40    CALL SIGMA(S2N1N,OUTPUT)
      SERAD=PSP*OUTPUT/AB(1)
      RETURN
C     FOLLOWING ARE DERIVATIVES OF FUNCTIONS
50    GO TO(60,70,80,90),CASE
C     THE VALUE OF SE2N+2'(Z,Q)
60    CALL SIGMA(DS2N2P,OUTPUT)
      SERAD=-PS*OUTPUT/AB(1)
      RETURN
C     THE VALUE OF SE2N+1'(Z,Q)
70    CALL SIGMA(DS2N1P,OUTPUT)
      SERAD=PS*OUTPUT/AB(1)
      RETURN
```

```
C     THE VALUE OF SE2N+2'(Z,-Q)
80    CALL SIGMA(DS2N2N,OUTPUT)
      SERAD=PSP*OUTPUT/AB(1)
      RETURN
C     THE VALUE OF SE2N+1'(Z,-Q)
90    CALL SIGMA(DS2N1N,OUTPUT)
      SERAD=PSP*OUTPUT/AB(1)
      RETURN
      END
      DOUBLE PRECISION FUNCTION S2N2P(K)
C     PURPOSE:  TO CALCULATE K TH TERM IN SUM OF SERIES
C               FOR SE2N+2(Z,Q).
      DOUBLE PRECISION AB(25),BSJV1,BSJV2,BSYV2,DBSJV1,DBSJV2,
     *                 DBSYV2
      COMMON/LOCAL/DUMMY1(8),AB
      COMMON/RADIAL/BSJV1(25),BSJV2(25),BSYV2(25),DBSJV1(25),
     *              DBSJV2(25),DBSYV2(25)
      S2N2P=AB(K)*(BSJV1(K)*BSJV2(K+2)-BSJV1(K+2)*BSJV2(K))
      IF(MOD(K,2).EQ.0)S2N2P=-S2N2P
      RETURN
      END
      DOUBLE PRECISION FUNCTION S2N1P(K)
C     PURPOSE:  TO CALCULATE K TH TERM IN SUM OF SERIES
C               FOR SE2N+1(Z,Q).
      DOUBLE PRECISION AB(25),BSJV1,BSJV2,BSYV2,DBSJV1,DBSJV2,
     *                 DBSYV2
      COMMON/LOCAL/DUMMY1(8),AB
      COMMON/RADIAL/BSJV1(25),BSJV2(25),BSYV2(25),DBSJV1(25),
     *              DBSJV2(25),DBSYV2(25)
      S2N1P=AB(K)*(BSJV1(K)*BSJV2(K+1)-BSJV1(K+1)*BSJV2(K))
      IF(MOD(K,2).EQ.0)S2N1P=-S2N1P
      RETURN
      END
      DOUBLE PRECISION FUNCTION S2N2N(K)
C     PURPOSE:  TO CALCULATE K TH TERM IN SUM OF SERIES
C               FOR SE2N(Z,-Q).
      DOUBLE PRECISION AB(25),BSIV1,BSIV2,BSKV2,DBSIV1,DBSIV2,
     *                 DBSKV2
      COMMON/LOCAL/DUMMY1(8),AB
      COMMON/RADIAL/BSIV1(25),BSIV2(25),BSKV2(25),DBSIV1(25),
     *              DBSIV2(25),DBSKV2(25)
      S2N2N=AB(K)*(BSIV1(K)*BSIV2(K+2)-BSIV1(K+2)*BSIV2(K))
      IF(MOD(K,2).EQ.0)S2N2N=-S2N2N
      RETURN
      END
      DOUBLE PRECISION FUNCTION S2N1N(K)
C     PURPOSE:  TO CALCULATE K TH TERM IN SUM OF SERIES
C               FOR SE2N+1(Z,-Q).
      DOUBLE PRECISION AB(25),BSIV1,BSIV2,BSKV2,DBSIV1,DBSIV2,
     *                 DBSKV2
      COMMON/LOCAL/DUMMY1(8),AB
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      COMMON/RADIAL/BSIV1(25),BSIV2(25),BSKV2(25),DBSIV1(25),
     *                DBSIV2(25),DBSKV2(25)
      S2N1N=AB(K)*(BSIV1(K)*BSIV2(K+1)-BSIV1(K+1)*BSIV2(K))
      IF(MOD(K,2).EQ.0)S2N1N=-S2N1N
      RETURN
      END
      DOUBLE PRECISION FUNCTION DS2N2P(K)
C     PURPOSE:  TO CALCULATE K TH TERM IN SUM OF SERIES
C               FOR SE2N+2'(Z,Q).
      DOUBLE PRECISION AB(25),BSJV1,BSJV2,BSYV2,DBSJV1,DBSJV2,
     *                 DBSYV2,V1,V2
      COMMON/LOCAL/DUMMY1(4),V1,V2,AB
      COMMON/RADIAL/BSJV1(25),BSJV2(25),BSYV2(25),DBSJV1(25),
     *                DBSJV2(25),DBSYV2(25)
      DS2N2P=AB(K)*(-DBSJV1(K)*BSJV2(K+2)*V1+
     *                 BSJV1(K)*DBSJV2(K+2)*V2+
     *       DBSJV1(K+2)*BSJV2(K)*V1-BSJV1(K+2)*DBSJV2(K)*V2)
      IF(MOD(K,2).EQ.0)DS2N2P=-DS2N2P
      RETURN
      END
      DOUBLE PRECISION FUNCTION DS2N1P(K)
C     PURPOSE:  TO CALCULATE K TH TERM IN SUM OF SERIES
C               FOR SE2N+1'(Z,Q).
      DOUBLE PRECISION AB(25),BSJV1,BSJV2,BSYV2,DBSJV1,DBSJV2,
     *                 DBSYV2,V1,V2
      COMMON/LOCAL/DUMMY1(4),V1,V2,AB
      COMMON/RADIAL/BSJV1(25),BSJV2(25),BSYV2(25),DBSJV1(25),
     *                DBSJV2(25),DBSYV2(25)
      DS2N1P=AB(K)*(-DBSJV1(K)*BSJV2(K+1)*V1+
     *                 BSJV1(K)*DBSJV2(K+1)*V2+
     *       DBSJV1(K+1)*BSJV2(K)*V1-BSJV1(K+1)*DBSJV2(K)*V2)
      IF(MOD(K,2).EQ.0)DS2N1P=-DS2N1P
      RETURN
      END
      DOUBLE PRECISION FUNCTION DS2N2N(K)
C     PURPOSE:  TO CALCULATE K TH TERM IN SUM OF SERIES
C               FOR SE2N+2'(Z,-Q).
      DOUBLE PRECISION AB(25),BSIV1,BSIV2,BSKV2,DBSIV1,DBSIV2,
     *                 DBSKV2,V1,V2
      COMMON/LOCAL/DUMMY1(4),V1,V2,AB
      COMMON/RADIAL/BSIV1(25),BSIV2(25),BSKV2(25),DBSIV1(25),
     *                DBSIV2(25),DBSKV2(25)
      DS2N2N=AB(K)*(-DBSIV1(K)*BSIV2(K+2)*V1+
     *                 BSIV1(K)*DBSIV2(K+2)*V2+
     *       DBSIV1(K+2)*BSIV2(K)*V1-BSIV1(K+2)*DBSIV2(K)*V2)
      IF(MOD(K,2).EQ.0)DS2N2N=-DS2N2N
      RETURN
      END
      DOUBLE PRECISION FUNCTION DS2N1N(K)
C     PURPOSE:  TO CALCULATE K TH TERM IN SUM OF SERIES
C               FOR SE2N+1'(Z,-Q).
```

```
      DOUBLE PRECISION AB(25),BSIV1,BSIV2,BSKV2,DBSIV1,DBSIV2,
     *                  DBSKV2,V1,V2
      COMMON/LOCAL/DUMMY1(4),V1,V2,AB
      COMMON/RADIAL/BSIV1(25),BSIV2(25),BSKV2(25),DBSIV1(25),
     *                 DBSIV2(25),DBSKV2(25)
      DS2N1N=AB(K)*(-DBSIV1(K)*BSIV2(K+1)*V1+
     *                 BSIV1(K)*DBSIV2(K+1)*V2+
     *        DBSIV1(K+1)*BSIV2(K)*V1-BSIV1(K+1)*DBSIV2(K)*V2)
      IF(MOD(K,2).EQ.0)DS2N1N=-DS2N1N
      RETURN
      END
      DOUBLE PRECISION FUNCTION FERAD(QD,R,DERIV,PS,AR,N)
C     PURPOSE:  TO COMPUTE A MODIFIED MATHIEU FUNCTION
C               (OR DERIVATIVE) OF SECOND KIND CORRESPONDING TO
C               EVEN MATHIEU FUNCTION ( CE FUNCTIONS)
      EXTERNAL FY2N,FY2N1,FK2N,FK2N1,DFY2N,DFY2N1,DFK2N,DFK2N1
      DOUBLE PRECISION AB(25),QD,PS,PSP,OUTPUT,AR(25),PI
      INTEGER R,CASE,DERIV
      COMMON/NTERM/N1
      COMMON/LOCAL/DUMMY(8),AB
      DATA PI/3.14159265358979300/
      N1=N
      PSP=PS
      DO 5 I=1,N
      AB(I)=AR(I)
5     CONTINUE
      CASE=MOD(R,2)+1
      IF(QD.LT.0.0D0) CASE=CASE+2
      IF(DERIV.EQ.1) GO TO 50
      GO TO(10,20,30,40),CASE
C     THE VALUE OF FEY2N(Z,Q)
10    CALL SIGMA(FY2N,OUTPUT)
      FERAD=PS*OUTPUT/AB(1)
      RETURN
C     THE VALUE OF FEY2N+1(Z,Q)
20    CALL SIGMA(FY2N1,OUTPUT)
      FERAD=PS*OUTPUT/AB(1)
      RETURN
C     THE VALUE OF FEK2N(Z,-Q)
30    CALL SIGMA(FK2N,OUTPUT)
      FERAD=PSP*OUTPUT/(AB(1)*PI)
      RETURN
C     THE VALUE OF FEK2N+1(Z,-Q)
40    CALL SIGMA(FK2N1,OUTPUT)
      FERAD=PSP*OUTPUT/(AB(1)*PI)
      RETURN
C     FOLLOWING ARE DERIVATIVES OF FUNCTIONS
50    GO TO(60,70,80,90),CASE
C     THE VALUE OF FEY2N'(Z,Q)
60    CALL SIGMA(DFY2N,OUTPUT)
      FERAD=PS*OUTPUT/AB(1)
```

```
      RETURN
C     THE VALUE OF FEY2N+1'(Z,Q)
70    CALL SIGMA(DFY2N1,OUTPUT)
      FERAD=PS*OUTPUT/AB(1)
      RETURN
C     THE VALUE OF FEK2N'(Z,-Q)
80    CALL SIGMA(DFK2N,OUTPUT)
      FERAD=PSP*OUTPUT/(AB(1)*PI)
      RETURN
C     THE VALUE OF FEK2N+1'(Z,-Q)
90    CALL SIGMA(DFK2N1,OUTPUT)
      FERAD=PSP*OUTPUT/(AB(1)*PI)
      RETURN
      END
      DOUBLE PRECISION FUNCTION FY2N(K)
C     PURPOSE:  TO CALCULATE K TH TERM IN SUM OF SERIES
C               FOR FEY2N(Z,Q).
      DOUBLE PRECISION AB(25),BSJV1,BSJV2,BSYV2,DBSJV1,DBSJV2,
     *                    DBSYV2
      COMMON/LOCAL/DUMMY1(8),AB
      COMMON/RADIAL/BSJV1(25),BSJV2(25),BSYV2(25),DBSJV1(25),
     *                 DBSJV2(25),DBSYV2(25)
      FY2N=AB(K)*BSJV1(K)*BSYV2(K)
      IF(MOD(K,2).EQ.0)FY2N=-FY2N
      RETURN
      END
      DOUBLE PRECISION FUNCTION FY2N1(K)
C     PURPOSE:  TO CALCULATE K TH TERM IN SUM OF SERIES
C               FEY2N+1(Z,Q).
      DOUBLE PRECISION AB(25),BSJV1,BSJV2,BSYV2,DBSJV1,DBSJV2,
     *                    DBSYV2
      COMMON/LOCAL/DUMMY1(8),AB
      COMMON/RADIAL/BSJV1(25),BSJV2(25),BSYV2(25),DBSJV1(25),
     *                 DBSJV2(25),DBSYV2(25)
      FY2N1=AB(K)*(BSJV1(K)*BSYV2(K+1)+BSJV1(K+1)*BSYV2(K))
      IF(MOD(K,2).EQ.0)FY2N1=-FY2N1
      RETURN
      END
      DOUBLE PRECISION FUNCTION FK2N(K)
C     PURPOSE:  TO CALCULATE K TH TERM IN SUM OF SERIES
C               FOR FEK2N(Z,Q).
      DOUBLE PRECISION AB(25),BSIV1,BSIV2,BSKV2,DBSIV1,DBSIV2,
     *                    DBSKV2
      COMMON/LOCAL/DUMMY1(8),AB
      COMMON/RADIAL/BSIV1(25),BSIV2(25),BSKV2(25),DBSIV1(25),
     *                 DBSIV2(25),DBSKV2(25)
      FK2N=AB(K)*BSIV1(K)*BSKV2(K)
      RETURN
      END
      DOUBLE PRECISION FUNCTION FK2N1(K)
C     PURPOSE:  TO CALCULATE K TH TERM IN SUM OF SERIES
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C                FOR FEK2N+1(Z,Q).
      DOUBLE PRECISION AB(25),BSIV1,BSIV2,BSKV2,DBSIV1,DBSIV2,
     *                 DBSKV2
      COMMON/LOCAL/DUMMY1(8),AB
      COMMON/RADIAL/BSIV1(25),BSIV2(25),BSKV2(25),DBSIV1(25),
     *              DBSIV2(25),DBSKV2(25)
      FK2N1=AB(K)*(BSIV1(K)*BSKV2(K+1)-BSIV1(K+1)*BSKV2(K))
      RETURN
      END
      DOUBLE PRECISION FUNCTION DFY2N(K)
C     PURPOSE:  TO CALCULATE K TH TERM IN SUM OF SERIES
C               FOR FEY2N'(Z,Q).
      DOUBLE PRECISION AB(25),BSJV1,BSJV2,BSYV2,DBSJV1,DBSJV2,
     *                 DBSYV2,V1,V2
      COMMON/LOCAL/DUMMY1(4),V1,V2,AB
      COMMON/RADIAL/BSJV1(25),BSJV2(25),BSYV2(25),DBSJV1(25),
     *              DBSJV2(25),DBSYV2(25)
      DFY2N=AB(K)*(-DBSJV1(K)*BSYV2(K)*V1+BSJV1(K)*DBSYV2(K)*V2)
      IF(MOD(K,2).EQ.0)DFY2N=-DFY2N
      RETURN
      END
      DOUBLE PRECISION FUNCTION DFY2N1(K)
C     PURPOSE:  TO CALCULATE K TH TERM IN SUM OF SERIES
C               FOR FEY2N+1'(Z,Q).
      DOUBLE PRECISION AB(25),BSJV1,BSJV2,BSYV2,DBSJV1,DBSJV2,
     *                 DBSYV2,V1,V2
      COMMON/LOCAL/DUMMY1(4),V1,V2,AB
      COMMON/RADIAL/BSJV1(25),BSJV2(25),BSYV2(25),DBSJV1(25),
     *              DBSJV2(25),DBSYV2(25)
      DFY2N1=AB(K)*(-DBSJV1(K)*BSYV2(K+1)*V1+
     *              BSJV1(K)*DBSYV2(K+1)*V2-
     *       DBSJV1(K+1)*BSYV2(K)*V1+BSJV1(K+1)*DBSYV2(K)*V2)
      IF(MOD(K,2).EQ.0)DFY2N1=-DFY2N1
      RETURN
      END
      DOUBLE PRECISION FUNCTION DFK2N(K)
C     PURPOSE:  TO CALCULATE K TH TERM IN SUM OF SERIES
C               FOR FEK2N'(Z,Q).
      DOUBLE PRECISION AB(25),BSIV1,BSIV2,BSKV2,DBSIV1,DBSIV2,
     *                 DBSKV2,V1,V2
      COMMON/LOCAL/DUMMY1(4),V1,V2,AB
      COMMON/RADIAL/BSIV1(25),BSIV2(25),BSKV2(25),DBSIV1(25),
     *              DBSIV2(25),DBSKV2(25)
      DFK2N=AB(K)*(-DBSIV1(K)*BSKV2(K)*V1+BSIV1(K)*DBSKV2(K)*V2)
      RETURN
      END
      DOUBLE PRECISION FUNCTION DFK2N1(K)
C     PURPOSE:  TO CALCULATE K TH TERM IN SUM OF SERIES
C               FOR FEK2N+1'(Z,-Q).
      DOUBLE PRECISION AB(25),BSIV1,BSIV2,BSKV2,DBSIV1,DBSIV2,
     *                 DBSKV2,V1,V2
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      COMMON/LOCAL/DUMMY1(4),V1,V2,AB
      COMMON/RADIAL/BSIV1(25),BSIV2(25),BSKV2(25),DBSIV1(25),
     *              DBSIV2(25),DBSKV2(25)
      DFK2N1=AB(K)*(-DBSIV1(K)*BSKV2(K+1)*V1+
     *              BSIV1(K)*DBSKV2(K+1)*V2+
     *       DBSIV1(K+1)*BSKV2(K)*V1-BSIV1(K+1)*DBSKV2(K)*V2)
      RETURN
      END
      DOUBLE PRECISION FUNCTION GERAD(QD,R,DERIV,PS,AR,N)
C     PURPOSE:  TO COMPUTE A MODIFIED MATHIEU FUNCTION
C               (OR DERIVATIVE) OF SECOND KIND CORRESPONDING TO
C               ODD MATHIEU FUNCTION ( CE FUNCTIONS)
      EXTERNAL GY2N2,GY2N1,GK2N2,GK2N1,DGY2N2,DGY2N1,DGK2N2,DGK2N1
      DOUBLE PRECISION AB(25),QD,PS,PSP,OUTPUT,AR(25),PI
      INTEGER R,CASE,DERIV
      COMMON/NTERM/N1
      COMMON/LOCAL/DUMMY(8),AB
      DATA PI/3.14159265358979300/
      N1=N
      PSP=PS
      DO 5 I=1,N
      AB(I)=AR(I)
5     CONTINUE
      CASE=MOD(R,2)+1
      IF(QD.LT.0.0D0) CASE=CASE+2
      IF(DERIV.EQ.1) GO TO 50
      GO TO(10,20,30,40),CASE
C     THE VALUE OF GEY2N+2(Z,Q)
10    CALL SIGMA(GY2N2,OUTPUT)
      GERAD=-PS*OUTPUT/AB(1)
      RETURN
C     THE VALUE OF GEY2N+1(Z,Q)
20    CALL SIGMA(GY2N1,OUTPUT)
      GERAD=PS*OUTPUT/AB(1)
      RETURN
C     THE VALUE OF GEK2N+2(Z,-Q)
30    CALL SIGMA(GK2N2,OUTPUT)
      GERAD=PSP*OUTPUT/(AB(1)*PI)
      RETURN
C     THE VALUE OF GEK2N+1(Z,-Q)
40    CALL SIGMA(GK2N1,OUTPUT)
      GERAD=PSP*OUTPUT/(AB(1)*PI)
      RETURN
C     FOLLOWING ARE DERIVATIVES OF FUNCTIONS
50    GO TO(60,70,80,90),CASE
C     THE VALUE OF GEY2N+2'(Z,Q)
60    CALL SIGMA(DGY2N2,OUTPUT)
      GERAD=-PS*OUTPUT/AB(1)
      RETURN
C     THE VALUE OF GEY2N+1'(Z,Q)
70    CALL SIGMA(DGY2N1,OUTPUT)
```

```
      GERAD=PS*OUTPUT/AB(1)
      RETURN
C     THE VALUE OF GEK2N+2'(Z,-Q)
80    CALL SIGMA(DGK2N2,OUTPUT)
      GERAD=PSP*OUTPUT/(AB(1)*PI)
      RETURN
C     THE VALUE OF GEK2N+1'(Z,-Q)
90    CALL SIGMA(DGK2N1,OUTPUT)
      GERAD=PSP*OUTPUT/(AB(1)*PI)
      RETURN
      END
      DOUBLE PRECISION FUNCTION GY2N2(K)
C     PURPOSE:  TO CALCULATE K TH TERM IN SUM OF SERIES
C               FOR GEY2N+2(Z,Q).
      DOUBLE PRECISION AB(25),BSJV1,BSJV2,BSYV2,DBSJV1,DBSJV2,
     *                 DBSYV2
      COMMON/LOCAL/DUMMY1(8),AB
      COMMON/RADIAL/BSJV1(25),BSJV2(25),BSYV2(25),DBSJV1(25),
     *              DBSJV2(25),DBSYV2(25)
      GY2N2=AB(K)*(BSJV1(K)*BSYV2(K+2)-BSJV1(K+2)*BSYV2(K))
      IF(MOD(K,2).EQ.0)GY2N2=-GY2N2
      RETURN
      END
      DOUBLE PRECISION FUNCTION GY2N1(K)
C     PURPOSE:  TO CALCULATE K TH TERM IN SUM OF SERIES
C               FOR GEY2N+1(Z,Q).
      DOUBLE PRECISION AB(25),BSJV1,BSJV2,BSYV2,DBSJV1,DBSJV2,
     *                 DBSYV2
      COMMON/LOCAL/DUMMY1(8),AB
      COMMON/RADIAL/BSJV1(25),BSJV2(25),BSYV2(25),DBSJV1(25),
     *              DBSJV2(25),DBSYV2(25)
      GY2N1=AB(K)*(BSJV1(K)*BSYV2(K+1)-BSJV1(K+1)*BSYV2(K))
      IF(MOD(K,2).EQ.0)GY2N1=-GY2N1
      RETURN
      END
      DOUBLE PRECISION FUNCTION GK2N2(K)
C     PURPOSE:  TO CALCULATE K TH TERM IN SUM OF SERIES
C               FOR GEK2N+2(Z,Q).
      DOUBLE PRECISION AB(25),BSIV1,BSIV2,BSKV2,DBSIV1,DBSIV2,
     *                 DBSKV2
      COMMON/LOCAL/DUMMY1(8),AB
      COMMON/RADIAL/BSIV1(25),BSIV2(25),BSKV2(25),DBSIV1(25),
     *              DBSIV2(25),DBSKV2(25)
      GK2N2=AB(K)*(BSIV1(K)*BSKV2(K+2)-BSIV1(K+2)*BSKV2(K))
      RETURN
      END
      DOUBLE PRECISION FUNCTION GK2N1(K)
C     PURPOSE:  TO CALCULATE K TH TERM IN SUM OF SERIES
C               FOR GEK2N+1(Z,-Q).
      DOUBLE PRECISION AB(25),BSIV1,BSIV2,BSKV2,DBSIV1,DBSIV2,
     *                 DBSKV2
```

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      COMMON/LOCAL/DUMMY1(8),AB
      COMMON/RADIAL/BSIV1(25),BSIV2(25),BSKV2(25),DBSIV1(25),
     *                   DBSIV2(25),DBSKV2(25)
      GK2N1=AB(K)*(BSIV1(K)*BSKV2(K+1)+BSIV1(K+1)*BSKV2(K))
      RETURN
      END
      DOUBLE PRECISION FUNCTION DGY2N2(K)
C     PURPOSE:  TO CALCULATE K TH TERM IN SUM OF SERIES
C               FOR GEY2N+2'(Z,Q).
      DOUBLE PRECISION AB(25),BSJV1,BSJV2,BSYV2,DBSJV1,DBSJV2,
     *                      DBSYV2,V1,V2
      COMMON/LOCAL/DUMMY1(4),V1,V2,AB
      COMMON/RADIAL/BSJV1(25),BSJV2(25),BSYV2(25),DBSJV1(25),
     *                   DBSJV2(25),DBSYV2(25)
      DGY2N2=AB(K)*(-DBSJV1(K)*BSYV2(K+2)*V1+
     *                    BSJV1(K)*DBSYV2(K+2)*V2+
     *        DBSJV1(K+2)*BSYV2(K)*V1-BSJV1(K+2)*DBSYV2(K)*V2)
      IF(MOD(K,2).EQ.0)DGY2N2=-DGY2N2
      RETURN
      END
      DOUBLE PRECISION FUNCTION DGY2N1(K)
C     PURPOSE:  TO CALCULATE K TH TERM IN SUM OF SERIES
C               FOR GEY2N+1'(Z,Q).
      DOUBLE PRECISION AB(25),BSJV1,BSJV2,BSYV2,DBSJV1,DBSJV2,
     *                      DBSYV2,V1,V2
      COMMON/LOCAL/DUMMY1(4),V1,V2,AB
      COMMON/RADIAL/BSJV1(25),BSJV2(25),BSYV2(25),DBSJV1(25),
     *                   DBSJV2(25),DBSYV2(25)
      DGY2N1=AB(K)*(-DBSJV1(K)*BSYV2(K+1)*V1+
     *                    BSJV1(K)*DBSYV2(K+1)*V2+
     *        DBSJV1(K+1)*BSYV2(K)*V1-BSJV1(K+1)*DBSYV2(K)*V2)
      IF(MOD(K,2).EQ.0)DGY2N1=-DGY2N1
      RETURN
      END
      DOUBLE PRECISION FUNCTION DGK2N2(K)
C     PURPOSE:  TO CALCULATE K TH TERM IN SUM OF SERIES
C               FOR GEK2N+2'(Z,-Q).
      DOUBLE PRECISION AB(25),BSIV1,BSIV2,BSKV2,DBSIV1,DBSIV2,
     *                      DBSKV2,V1,V2
      COMMON/LOCAL/DUMMY1(4),V1,V2,AB
      COMMON/RADIAL/BSIV1(25),BSIV2(25),BSKV2(25),DBSIV1(25),
     *                   DBSIV2(25),DBSKV2(25)
      DGK2N2=AB(K)*(-DBSIV1(K)*BSKV2(K+2)*V1+
     *                    BSIV1(K)*DBSKV2(K+2)*V2+
     *        DBSIV1(K+2)*BSKV2(K)*V1-BSIV1(K+2)*DBSKV2(K)*V2)
      RETURN
      END
      DOUBLE PRECISION FUNCTION DGK2N1(K)
C     PURPOSE:  TO CALCULATE K TH TERM IN SUM OF SERIES
C               FOR GEK2N+1'(Z,-Q).
      DOUBLE PRECISION AB(25),BSIV1,BSIV2,BSKV2,DBSIV1,DBSIV2,
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```
     *              DBSKV2,V1,V2
      COMMON/LOCAL/DUMMY1(4),V1,V2,AB
      COMMON/RADIAL/BSIV1(25),BSIV2(25),BSKV2(25),DBSIV1(25),
     *              DBSIV2(25),DBSKV2(25)
      DGK2N1=AB(K)*(-DBSIV1(K)*BSKV2(K+1)*V1+
     *              BSIV1(K)*DBSKV2(K+1)*V2-
     *      DBSIV1(K+1)*BSKV2(K)*V1+BSIV1(K+1)*DBSKV2(K)*V2)
      RETURN
      END
```

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ANISOTROPIC ELLIPTIC OPTICAL FIBERS

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The exact characteristic equation for an anisotropic elliptic optical fiber is obtained for odd and even hybrid modes in terms of infinite determinants utilizing Mathieu and modified Mathieu functions. A simplified characteristic equation is obtained by applying the weakly guiding approximation such that the difference in the refractive indices of the core and the cladding is small.

The simplified characteristic equation is used to compute the normalized guide wavelength for an elliptical fiber. When the anisotropic parameter is equal to unity, the results are compared with the previous research and they are in close agreement.

For a fixed value of normalized cross-section area or major axis, the normalized guide wavelength λ/λ_o for an anisotropic elliptic fiber is small for larger the value of anisotropy. This condition indicates that more energy is carried inside of the fiber. However, the geometry and anisotropy of the fiber have a smaller effect when the normalized cross-section area is very small or very large.